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REAL, COMPLEX AND FUNCTIONAL ANALYSIS

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RUC PROPERTY FOR CHAOS OF RANDOM VARIABLES IN THE UNIFORM NORM

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Abstract. Let $X = \{X_k\}_{k=1}^{\infty}$ be a sequence of independent symmetric bounded random variables. This paper investigates systems of the form $\{X_iX_j\}_{i<j}$, $\{X_iX_jX_k\}_{i<j<k}, \ldots$, finite unions of such systems, and systems close to them, in the space L_∞ of bounded random variables. Series over such systems do not hold the property of unconditionality: the convergence of the series depends on the ordering of the terms. At the same time, as we demonstrate in the paper, such systems posess a very close property of random unconditional convergence (or RUC-property).

СВОЙСТВО RUC ДЛЯ ХАОСА СЛУЧАЙНЫХ ВЕЛИЧИН В РАВНОМЕРНОЙ НОРМЕ

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Ключевые слова: равномерная норма, случайная безусловная сходимость, геометрия банаховых пространств, хаос Радемахера, полиномиальный хаос, симметричные случайные величины.

Аннотация. Пусть $X = \{X_k\}_{k=1}^{\infty}$ – последовательность независимых симметричных и ограниченных случайных величин. В работе рассматриваются системы вида $\{X_iX_j\}_{i<j}$, $\{X_iX_jX_k\}_{i<j<k}, \ldots$, конечные объединения таких систем и близкие к ним системы в пространстве L_∞ ограниченных случайных величин. Ряды по таким системам не обладают свойством безусловности: сходимость рядов зависит от порядка, в котором нумеруются элементы системы. В то же время, как показано в работе, такие системы обладают очень бизким свойством случайной безусловной сходимости.

1. Introduction

Investigating the behavior of special sequences is a cornerstone of geometric Banach space theory [1; 2]. The properties associated with random sequences and series are particularly important [3; 4]. The simplest version of such random constructions arises by applying random signs to the terms of a series and studying norm changes of the sum under such arrangements. Another probabilistic method can be used when the Banach space itself consists of random variables, such as the Lebesgue space of measurable functions on the interval. Here, one studies sequences of independent random variables or polynomial forms from such sequences [5–11]. The independence of sequence elements allows for the application of general and strong results for sums over such terms, related to distribution estimates, moments, and limit theorems. At the same time, these sequences provide a rich source of examples and counterexamples that illuminate the geometry of the underlying space. By considering sums in Banach spaces of random variables with random coefficients, we can combine these two approaches of applying probabilistic methods to study the geometry of subspaces in such spaces.

We follow papers [12; 13], which initiated the study of sums over Rademacher chaos within the space $L_\infty[0,1]$. This space is viewed as the set of bounded random variables on the unit interval with the Lebesgue measure. The authors investigated the stability properties of norms for such sums under a random arrangement of signs. Let us recall the basic concepts and formulate some results from these works. Rademacher functions $r_k(t)$, for $t \in [0,1]$ and $k \in \mathbb{N}$, can be defined as follows:

$$r_k(t) = (-1)^{[2^k t]},\ k = 1,2,...,$$

where $[x]$ denotes the integer part of the number $[x]$. Rademacher functions are used in a large number of fundamental and applied problems[14–17]. The following fact was proved in [13]. For any $n \in \mathbb{N}$ and any real coefficients $a_{i,j}$, $1 \leqslant i < j \leqslant n$, it holds that

$$\begin{aligned} \mathsf{E}_\theta \left\| \sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i r_j \right\|_{L_\infty([0,1])} &\asymp \min_{\theta_{i,j}=\pm 1} \left\| \sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i r_j \right\|_{L_\infty([0,1])} \asymp \\ &\asymp \max \left\{ \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} a_{ij}^2 \right)^{1/2}, \sum_{j=2}^{n} \left(\sum_{i=1}^{j-1} a_{ij}^2 \right)^{1/2} \right\}. \end{aligned} \tag{1}$$

Here, $r_i = r_i(t)$ are Rademacher functions, θ_{ij} are independent signs (i. e., ± 1 valued random variables), and E_θ denotes the expectation with respect to these signs. The notation $X \asymp Y$ means that $c_1 Y \leqslant X \leqslant c_2 Y$ for some universal constants $c_1, c_2 > 0$. This result establishes *the random unconditional convergence (RUC) property* for the second-degree Rademacher chaos in L_∞ and connects its norm to one special norm of the coefficient matrix. RUC property was introduced by Billard, Kwapien, Pelczynski and Samuel in [18]. It shows that although the system may not be an unconditional basic sequence (basis), there holds a certain relaxation.

The nature of Rademacher random variables (we then use term rvs) gives the idea that results concerning it can be extended to similar random variables, such as symmetric bounded random variables. Moreover, the identical distribution of such rvs is not necessary for properties under investigation. A primary objective of this work is to extend the aforementioned L_∞-norm equivalences and the RUC property to polynomial chaos constructed from sequences $(X_1, X_2, \ldots, X_n, \ldots)$ of real-valued independent symmetric random variables with $\|X_i\|_{L_\infty} = C_i > 0$. We demonstrate that these extensions hold, with the key modification being a rescaling of the chaos coefficients by the respective bounding constants C_i. In addition, the paper shows that chaoses of different degrees can be combined while maintaining the property of random unconditional convergence.

The paper is organized as follows.

In Section 2 we present general definitions, some results from previous works that we will rely on, and auxiliary statements.

In Section 3 we consider systems formed by mixing the first- and second-degree Rademacher chaos. We examine two variants of such mixing. The first, more simple variant uses three independent copies of the Rademacher sequences $\{r_k\}, \{r'_i\}, \{r'_j\}$ and examines the behavior in L_∞ of sums of the form

$$S_{\text{sep}}(t) = \sum_{k=1}^{n} b_k r_k(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} r'_i(t) r''_j(t).$$

The index "sep" in S_{sep} means that we are considering *separated* (or *decoupled*) chaos, i. e. chaos constructed from independent copies of the original sequence of independent random variables. In the second case, we work with ordinary (or *unseparated*) Rademacher chaos, i. e., we study the behavior of sums of the form

$$S(t) = \sum_{k=1}^{n} b_k r_k(t) + \sum_{1\leqslant i<j\leqslant n} a_{ij} r_i(t) r_j(t).$$

The key property that allows us to transfer the results for homogeneous chaos from papers [12; 13] to the mixed chaos we consider is the complementedness of homogeneous chaos in mixed chaos. This property can also be obtained from the work of [19]. We, however, also consider a direct proof of the complementedness property, which is especially simple in the considered case of first- and second-degree chaos.

In Section 4 we extend the results of Section 3 to systems constructed from a sequence of independent symmetric bounded random variables, not necessarily identically distributed. The main idea is that the subspaces $\mathcal{X} := \text{span}[\{X_k\}, \{X_i X_j\}]$ and $\mathcal{Y} := \text{span}[\{Y_k\}, \{Y_i Y_j\}]$ generated by different systems of independent random variables are isometric to each other in the case of $\|X_k\|_{L_\infty} = \|Y_k\|_{L_\infty}, k = 1,2\ldots$:

$$\left\|\sum_{k=1}^{n} b_k X_k + \sum_{1\leqslant i<j\leqslant n} a_{ij} X_i X_j\right\|_{L_\infty} = \left\|\sum_{k=1}^{n} b_k Y_k + \sum_{1\leqslant i<j\leqslant n} a_{ij} Y_i Y_j\right\|_{L_\infty},$$

and the same equalities are valid for chaos of arbitrary degree. Formally, we prove this equality for homogeneous separated chaos. The result is then extended to the unseparated chaos via the decoupling method and finally to the mixed chaos by using the complementedness of homogeneous parts.

Rademacher chaos, discussed in Section 3, is a special case of the more general chaos studied in Section 4. Moreover, results for the general case can be proved independently of Rademacher chaos. However, we stress the case of the Rademacher chaos due to its particular importance for applications. Bilinear and quadratic binary forms, equivalent to separated and unseparated Rademacher chaos, respectively, are important in neural network models of associative memory [20–22], energy analysis of spin glasses [23; 24], and adiabatic quantum computing [25].

2. Preliminaries and auxiliary results

A sequence $\{x_k\}_{k=1}^{\infty}$ of elements in a Banach space X is called *basic* if it is a Schauder basis for its closed linear span $\overline{\text{span}}\{x_k\}$. A basic sequence $\{x_k\}$ is an *unconditional basic sequence* if for any $x = \sum_k a_k x_k \in \overline{\text{span}}\{x_k\}$ and any sequence of signs $\epsilon_k = \pm 1$, the series $\sum_k \epsilon_k a_k x_k$ converges. In this case there exists a constant $C_u \geqslant 1$, not dependent on x, such that

$$\left\|\sum_k \epsilon_k a_k x_k\right\|_X \leqslant C_u \left\|\sum_k a_k x_k\right\|_X.$$

The elements of the unconditional basis sequence form a basis in $\overline{\text{span}}\{x_k\}$ under any permutation. This is equivalent to the property of convergence of series for all arrangements of signs, indicated in our definition of unconditionality. For basic and unconditional basic sequences we refer to [2]. Note that we also use the "inverse" form of the previous inequality

$$\left\|\sum_k a_k x_k\right\|_X \leqslant C_u \left\|\sum_k \epsilon_k a_k x_k\right\|_X.$$

Equivalence follows since both inequalities must be valid for any a_k and ϵ_k.

It is known that the Rademacher system $\{r_k\}$, as well as systems consisting of products of Rademacher functions $\{r_i r_j\}$, $\{r_i r_j r_k\}$ ..., is an unconditional basic sequence in $L_p([0,1])$ for $1 \leqslant p < \infty$ [26]. It is obvious that the Rademacher system will retain the property of unconditionality in the space $L_\infty[0,1]$, since the distribution of this system does not change when its elements are rearranged. However, this is not the case for the system of products [27; 28].

We follow ([13, Remark 1], [18]) to give the following definition. A sequence of elements $\{x_k\}$ in a Banach space X is said to possess the *Random Unconditional Convergence (RUC) property* if there exist universal constants such that for any finite sequence of scalars $\{a_k\}$, $1 \leqslant k \leqslant n$,

$$\mathsf{E}_\theta \left\|\sum_{k=1}^{n} \theta_k a_k x_k\right\|_X \asymp \min_{\theta_k=\pm 1} \left\|\sum_{k=1}^{n} \theta_k a_k x_k\right\|_X.$$

where $\{\theta_k\}$ is a sequence of independent Rademacher signs, i. e. for the probabilities of values of random variables θ_k the condition $\mathsf{P}\{\theta_k = 1\} = \mathsf{P}\{\theta_k = -1\} = 1/2$ is satisfied. This shows that the expectation of the norm behaves like the minimum, so they are "close". We note that in definition of the RUC property we consider finite sums only and consequently the order of elements of the sequence does not matter.

We consider Rademacher chaos polynomials. A d-th degree *homogeneous unseparated Rademacher chaos* (or *homogeneous Rademacher chaos*) is a system consisting of functions of variable $t \in [0,1]$ of the form

$$(r_{j_1} \ldots r_{j_d})(t) = r_{j_1}(t) \ldots r_{j_d}(t), \quad j_1 < j_2 < \ldots < j_d.$$

We then consider polynomials constructed from these functions of the form

$$P(t)=\sum_{1\leqslant j_1<j_2<\cdots<j_d\leqslant n} a_{j_1,\dots,j_d} r_{j_1}(t)\dots r_{j_d}(t),$$

where $a_{j_1,\dots,j_d}$ are real coefficients. We will call such functions *homogeneous Rademacher chaos polynomials*. The *homogeneous multiple Rademacher system of degree d* (also referred to as separated or decoupled Rademacher chaos of d-th degree) consists of functions of d variables $(t_1,\dots,t_d)\in[0,1]^d$:

$$(r_{j_1}\otimes\cdots\otimes r_{j_d})(t_1,\dots,t_d)=r_{j_1}(t_1)\dots r_{j_d}(t_d).$$

A linear combination of such elements,

$$P_{\text{sep}}(t_1,\dots,t_d)=\sum_{1\leqslant j_1,\dots,j_d\leqslant n} a_{j_1,\dots,j_d} r_{j_1}(t_1)\dots r_{j_d}(t_d),$$

we will call a d-th degree *homogeneous multiple Rademacher system polynomial*.

The L_∞-norm of a function $f:[0,1]^d\to\mathbb{R}$ is $||f||_{L_\infty}=\sup_{(t_1,\dots,t_d)\in[0,1]^d}|f(t_1,\dots,t_d)|$. For d-th degree multiple Rademacher system polynomial $P_{\text{sep}}(t_1,\dots,t_d)$, this is equivalent to the maximum over all 2^n sign combinations $\epsilon=(\epsilon_1,\dots,\epsilon_n)\in\{\pm1\}^n$:

$$\|P_{\text{sep}}\|_{L_\infty}=\max_{\epsilon\in\{\pm1\}^n}\left|\sum_{1\leqslant j_1,\dots,j_d\leqslant n} a_{j_1,\dots,j_d}\epsilon_{j_1}\dots\epsilon_{j_d}\right|.$$

Analagous relation holds for Rademacher chaos polynomial, but signs may dependent in that case.

We will need the following decoupling argument.

Lemma 2.1 (Decoupling for L_∞-norms, cf. [13, Corollary 1; 29, Theorem 3.1.1]). *Let $d,n\in\mathbb{N}$ with $d\leqslant n$. Let $(\xi_1,\dots,\xi_n)$ be a sequence of bounded independent random variables, and let $(\xi_1^{(k)},\dots,\xi_n^{(k)})$, for $k=1,\dots,d$, be d independent copies of this sequence. Suppose that coefficients $d_{j_1,\dots,j_d}$ are symmetric, i. e. $d_{j_1,\dots,j_d}=d_{j_{\pi(1)},\dots,j_{\pi(d)}}$ for each multi-index $(j_1,\dots,j_d)\in\tilde{N}_n^d:=\{(i_1,\dots,i_d)\in\{1,\dots,n\}^d: i_p\neq i_q \text{ if } p\neq q\}$ and every permutation π of $\{1,\dots,d\}$. Then,*

$$c_d\left\|\sum_{(i_1,\dots,i_d)\in\tilde{N}_n^d} d_{i_1,\dots,i_d}\xi_{i_1}^{(1)}\dots\xi_{i_d}^{(d)}\right\|_{L_\infty(\Omega_1\times\ldots\times\Omega_d)}\leqslant\left\|\sum_{(i_1,\dots,i_d)\in\tilde{N}_n^d} d_{i_1,\dots,i_d}\xi_{i_1}\dots\xi_{i_d}\right\|_{L_\infty(\Omega)}\leqslant$$
$$\leqslant\left\|\sum_{(i_1,\dots,i_d)\in\tilde{N}_n^d} d_{i_1,\dots,i_d}\xi_{i_1}^{(1)}\dots\xi_{i_d}^{(d)}\right\|_{L_\infty(\Omega_1\times\ldots\times\Omega_d)},$$

where c_d is constant depending only on d, and the L_∞-norms are essential suprema over the respective probability spaces Ω (for ξ_k) and $\Omega_1\times\ldots\times\Omega_d$ (for $\xi_k^{(j)}$).

Note that the right inequality in Lemma 2.1 is elementary: the set of essential values of the random variable $\sum d_{i_1,\dots,i_d}\xi_{i_1}\dots\xi_{i_d}$ is included in the set of essential values of the random variable $\sum d_{i_1,\dots,i_d}\xi_{i_1}^{(1)}\dots\xi_{i_d}^{(d)}$.

Let $d,n\in\mathbb{N}$ with $1\leqslant d\leqslant n$. Let $\mathbb{N}_n^d$ be the set of multi-indices $J=(j_1,\dots,j_d)$ such that $j_k\in[n]$, where $[n]:=\{1,2,\dots,n\}$. For $k\in\{1,\dots,d\}$, let J_k' denote the multi-index $(j_1,\dots,j_{k-1},j_{k+1},\dots,j_d)$, and also denote $t_k'=(t_1,\dots,t_{k-1},t_{k+1},\dots,t_d)$. The *multiple Rademacher system of degree d* is $\{r_J^{\otimes}\}_{J\in\mathbb{N}_n^d}$, where $r_J^{\otimes}(t_1,\dots,t_d)=r_{j_1}(t_1)\dots r_{j_d}(t_d)$.

Then we define Δ^d be the set of multi-indices $J=(j_1,\dots,j_d)$ such that $1\leqslant j_1<j_2<\cdots<j_d$. *The (homogeneous) Rademacher chaos of degree d* is a function $\{\mathbf{r}_J\}_{J\in\Delta^d}$, where $\mathbf{r}_J(t)=r_{j_1}(t)\dots r_{j_d}(t)$, $t\in[0,1]$. By Δ_n^d we denote the set $\{J=(j_1,j_2,\dots,j_d): 1\leqslant j_1<j_2<\ldots<j_d\leqslant n\}$.

Also we use elements of the multiple Rademacher system of the form

$$r_{J_k'}^{\otimes}(t_k')=r_{j_1}(t_1)\dots r_{j_{k-1}}(t_{k-1})r_{j_{k+1}}(t_{k+1})\dots r_{j_d}(t_d).$$

Finally, for every $d,n\in\mathbb{N},k=1,2,\dots,d$ and $l=1,2,\dots,n$ we put

$$\mathbb{N}_n^d(k,l)=\{J=(j_1,\dots,j_d)\in\mathbb{N}_n^d: j_k=l\}.$$

Now we discuss the two central theorems for our paper. They establish the RUC property of the multiple Rademacher system and Rademacher chaos of degree d in L_∞.

Theorem 2.2 [13, Theorem 4]. *For every $d \in \mathbb{N}$ the multiple Rademacher system $\{r_J^{\otimes}\}_{J\in\mathbb{N}^d}$ has the RUC property in $L_\infty([0,1]^d)$. More precisely, for all $n \in \mathbb{N}$ and $a_J \in \mathbb{R}^d, J \in \mathbb{N}_n^d$ the following inequalities hold:*

$$\left\| \sum_{J\in\mathbb{N}_n^d} a_J r_J^{\otimes} \right\|_{L_\infty([0,1]^d)} \geqslant 2^{\frac{1-d}{2}} \max_{k\in[d]} \sum_{l=1}^{n} \left(\sum_{J\in\mathbb{N}_n^d(k,l)} a_J^2 \right)^{1/2}, \tag{2}$$

and

$$\mathsf{E}_\theta \left\| \sum_{J\in\mathbb{N}_n^d} a_J \theta_J r_J^{\otimes} \right\|_{L_\infty([0,1]^d)} \leqslant \sum_{k=1}^{d} 2^{k-1} \sum_{l=1}^{n} \left(\sum_{J\in\mathbb{N}_n^d(k,l)} a_J^2 \right)^{1/2}, \tag{3}$$

where $(\theta_J)_{J\in\mathbb{N}_n^d}$ is a system of independent random signs, i. e. $\mathsf{P}\{\theta_J = 1\} = \mathsf{P}\{\theta_J = -1\} = 1/2$, $J \in \mathbb{N}_n^d$.

Theorem 2.3 [13, Corollary 7]. *Let $d, n \in \mathbb{N}$, $d \leqslant n$. There exist universal constant C_d' (depending only on d) such that for any real coefficients $(a_J)_{J\in\Delta_n^d}$,*

$$\min_{\theta} \left\| \sum_{J\in\Delta_n^d} \theta_J a_J \boldsymbol{r}_J \right\|_{L_\infty([0,1])} \leqslant \mathsf{E}_\theta \left\| \sum_{J\in\Delta_n^d} \theta_J a_J \boldsymbol{r}_J \right\|_{L_\infty([0,1])} \leqslant C_d' \left\| \sum_{J\in\Delta_n^d} a_J \boldsymbol{r}_J \right\|_{L_\infty([0,1])}, \tag{4}$$

where $(\theta_J)_{J\in\Delta_n^d}$ is a sequence of independent random signs.

Let us briefly describe the main ideas from [13] used in proving these results. We consider the case $d = 2$. For the lower bound on $\|\sum_{i=1}^n \sum_{j=1}^n \theta_{i,j} a_{i,j} r_i \otimes r_j\|_{L_\infty([0,1]^2)}$, one can use Szarek's refinement of Khintchine's inequality for L_1-norms [30]. We choose t_1, argument of the first function of products $r_i \otimes r_j = r_i(t_1) r_j(t_2)$, in an appropriate way, and the problem is reduced to estimating the L_1-norm of a Rademacher sum of degree 1 with respect to the remaining variable. Applying Khintchine's inequality then yields a lower bound in terms of $L_{2,1}$-norm:

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} r_i \otimes r_j \right\|_{L_\infty([0,1]^2)} \geqslant \sum_{i=1}^{n} \int_0^1 \left| \sum_{j=1}^{n} a_{ij} r_j(t) \right| dt \geqslant \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

The L_∞-norm of left hand side of (2) is thus bounded below. As we have symmetry in indices i and j, swapping them, we get another lower bound. For the upper bound (3) explanation authors use such techniques as the symmetrization trick and Ledoux–Talagrand contraction principle. It should be noted that the specific method of applying these techniques to obtain the upper bound was taken from paper [31]. For more thorough explanations we refer to [13]. Now, having these estimates and using Lemma 2.1, we proceed to RUC property for Rademacher chaos, i. e. (4).

We will consider multilinear and polynomial forms constructed from systems of random variables, which are defined on a probability space $([0,1], \mu)$ with standard Lebesgue measure, or on products of such probability spaces. It is easy to see that the main results remain valid when replacing the segment $[0,1]$ with an arbitrary probability space.

Let us agree on the terminology used.

Let $X = (X_k)$ be a sequence of independent random variables, and $X^{(1)} = (X_k^{(1)})$, $X^{(2)} = (X_k^{(2)})$, ..., $X^{(d)} = (X_k^{(d)})$ be its independent copies. This means that the systems $X, X^{(1)}, X^{(2)}, \dots, X^{(d)}$ are identically distributed and independent in the aggregate. We will call the system $\{X_{j_1}^{(1)} X_{j_2}^{(2)} \dots X_{j_d}^{(d)}\}_{(j_1,j_2,\dots,j_d)\in\mathbb{N}^d}$ a *homogeneous multiple random system of degree d*, and the union of such homogeneous and mutually independent systems of degrees $1, 2, \dots, d$ – a *mixed multiple random system of degree d*.

We will also consider systems generated by a single sequence X, without using its independent copies. We will call the system $\{X_{j_1} X_{j_2} \dots X_{j_d}\}_{(j_1,j_2,\dots,j_d)\in\Delta^d}$ a *homogeneous chaos of degree d*, and the union of such homogeneous systems of degrees $1, 2, \dots, d$ – a *mixed chaos of degree d*.

Thus, the homogeneous multiple Rademacher system and homogeneous Rademacher chaos defined above, which appear in Theorems 2.2 and 2.3, respectively, turn out to be special cases of a homogeneous multiple random system and homogeneous chaos. We note that the precise ordering of elements of these systems is not relevant for the RUC-property discussed in the article. However, it should be noted that such systems will form basic sequences if they are numbered using the lexicographic order on the index set [32].

Next, we will work with polynomials, by which we mean finite linear combinations of some elements of the introduced system. To specify the underlying system for a given polynomial, we will use a corresponding prefix. For example, a 3-degree homogeneous chaos polynomial will look like this:

$$P_3(X) = \sum_{1\leqslant i<j<k\leqslant n} a_{ijk} X_i X_j X_k.$$

And a 2-degree mixed multiple random system polynomial will look like this:

$$S_{\mathrm{sep}}(X^{(1)}, X^{(2)}, X^{(3)}) = \sum_{k=1}^{n} b_k X_k^{(1)} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} X_i^{(2)} X_j^{(3)}.$$

Note also that Lemma 2.1 involves both a homogeneous chaos polynomial and a homogeneous multiple random system polynomial. The distributions of these polynomials are different, but the lemma shows that their L_∞-norms are equivalent. Along with the notations $P_3(X), S_{\mathrm{sep}}(X^{(1)}, X^{(2)}, X^{(3)})$, as above, in which we emphasize the dependence of our polynomials on a system of independent rvs, we will also use notations of the form $P_3(t), S_{\mathrm{sep}}(t_1,t_2,t_3)$, in which we consider our polynomials as random variables (functions) of the variables $t \in [0,1]$, $(t_1,t_2,t_3) \in [0,1]^3$.

It is also worth noting that every mixed chaos polynomial $Q(X)$ of degree d can be uniquely represented as the sum of its homogeneous parts:

$$Q(X) = \sum_{k=1}^{d} Q_k(X),$$

where $Q_k(X)$ is a k-th degree homogeneous chaos polynomial (or the k-th homogeneous component of Q). An important result of Kwapien [19, Lemma 2] states that the mean values of a mixed chaos polynomial $Q(X)$ constructed from a symmetric vector X dominate the mean values of its homogeneous components.

Lemma 2.4 (Kwapien, [19, Lemma 2]) . *Let F be a vector space, and let $\varphi : F \to \mathbb{R}^+$ be a convex function such that $\varphi(-x) = \varphi(x)$ for all $x \in F$. Let $Q(\eta)$ be a mixed chaos polynomial of degree d with coefficients in F, where $\eta = (\eta_1, \ldots, \eta_n)$ is a vector of independent symmetric random variables. Let $Q_k(\eta)$ denote its k-th homogeneous component, for $1 \leqslant k \leqslant d$. Then there exists a constant K_d, depending only on d, such that*

$$\mathsf{E}\left[\varphi(Q_k(\eta))\right] \leqslant \mathsf{E}\left[\varphi(K_d Q(\eta))\right].$$

We will use the following corollary.

Corollary 2.5. *There is a constant K_d, depending only on d, such that for every mixed chaos polynomial $Q(\eta)$ of degree d, for every homogeneous component $Q_k(\eta)$ of this polynimial and every vector $\eta = (\eta_1, \ldots, \eta_n)$ of independent bounded symmetric random variables we have*

$$\|Q_k(\eta)\|_{L_\infty} \leqslant K_d \|Q(\eta)\|_{L_\infty}. \tag{5}$$

Proof. The function $\varphi(x) = |x|^p$ for $x \in [0,1]$ satisfies all conditions of Lemma 2.4. Applying the lemma and taking the p-th root, we have

$$\left(\mathsf{E}|Q_k(\eta)|^p\right)^{1/p} \leqslant K_d \left(\mathsf{E}|Q(\eta)|^p\right)^{1/p}.$$

Passing to the limit as $p \to \infty$, this yields the L_∞ estimate (5). □

Remark 2.6. *It is known that K_d can be taken as 2^d, which is also cited by Kwapien.*

This paper considers chaoses constructed from a sequence of independent symmetric bounded random variables, which we will denote as $(X_k)_{k=1}^{\infty}$, such that for each k, $\|X_k\|_{L_\infty} = C_k > 0$.

3. RUC property for mixed multiple Rademacher system and mixed Rademacher chaos

In this section we extend the results of papers [12; 13] about homogeneous Rademacher chaos to the case of mixed Rademacher chaos. Thus we will consider 2-th degree polynomials of the form

$$S(t) = \sum_{k=1}^{n} b_k r_k(t) + \sum_{1\leqslant i<j\leqslant n} a_{ij} r_i(t) r_j(t), \tag{6}$$

where b_k and a_{ij} are real coefficients. We denote the first-degree homogeneous part of $S(t)$ as $P_1(t)$ and the second-degree part as $P_2(t)$, so $S(t) = P_1(t) + P_2(t)$.

We also consider "separated" version of chaos, in which different degree terms are generated by independent copies of Rademacher sequences. More precisely, we will consider mixed multiple Rademacher system polinomials of the form

$$S_{\text{sep}}(t) = \sum_{k=1}^{n} b_k r_k(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} r_i'(t) r_j''(t), \tag{7}$$

where r, r', and r'' are three mutually independent Rademacher sequences. Note that this function has the same distribution as following function, defined on $[0,1]^3$:

$$S_{\text{sep}}(t_0,t_1,t_2) = \sum_{k=1}^{n} b_k r_k(t_0) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} r_i(t_1) r_j(t_2). \tag{8}$$

We will use this fact of equivalence of distributions in our proofs.

We first consider the simpler case of mixed multiple Rademacher system, where the components of different degrees are generated by independent Rademacher sequences.

3.1. RUC property for mixed multiple Rademacher system

Proposition 3.1. *Let $S_{\text{sep}}(t) = P_1(t) + P_2(t)$ be a mixed Rademacher multiple system polynomial of second degree, where*

$$P_1(t) = \sum_{k=1}^{n} b_k r_k(t), \quad P_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} r_i'(t) r_j''(t).$$

Then

$$\|S_{\text{sep}}\|_{L_\infty[0,1]} = \|P_1\|_{L_\infty[0,1]} + \|P_2\|_{L_\infty[0,1]}. \tag{9}$$

Proof. We use equimeasurability of $S_{\text{sep}}(t)$ and $S_{\text{sep}}(t_1,t_2,t_3)$, which follows from (7) and (8). We know that for mixed Rademacher system polynomial of second degree their L_∞-norm is the absolute value of their sum for certain signs arrangement, i. e. there exists a sign configurations $(\epsilon_k, \epsilon_i', \epsilon_j'') \in \in \{-1,1\}^n \times \{-1,1\}^n \times \{-1,1\}^n$ which corresponds to such t_0^*, t_1^*, t_2^*, where maximum is attained, such that:

$$\|P_1\|_{L_\infty([0,1])} = \max_{t_0\in[0,1]} |\sum_{k=1}^{n} b_k r_k(t_0)| = \max_{\epsilon_k} |\sum_{k=1}^{n} b_k \epsilon_k|$$

and

$$\|P_2\|_{L_\infty([0,1]^2)} = \max_{t_1,t_2\in[0,1]^2} |\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} r_i(t_1) r_j(t_2)| = \max_{\epsilon_i',\epsilon_j''} |\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\epsilon_i'\epsilon_j''|.$$

These maxima we denote correspondingly by M_1 and M_2. Let also s_1 and s_2 denote the sign of the sum under the module in points t_0^*, t_1^*, t_2^*. Now we consider the symmetry argument which later be modified to the Rademacher chaos case. Because we choose t_0 independently of t_1, t_2, we can always have $s := s_1 = s_2$. Indeed, if these signs are different, we just take t_0^{**} such that $r_k(t_0^{**}) = -r_k(t_0^*)$ for all k. Such point always exists, because it corresponds to $(-\epsilon_k)$ sequence of signs. Therefore, we change the sign of P_1 without changing its absolute value. Therefore, we have

$$|P_1(t_0^*) + P_2(t_1^*, t_2^*)| = |sM_1 + sM_2| = |s(M_1+M_2)| = M_1 + M_2.$$

Taking maxima of both sides of the equation, we get

$$\|S_{\text{sep}}\|_{L_\infty} = \max_{t_0,t_1,t_2} |P_1(t_0) + P_2(t_1,t_2)| \geqslant M_1 + M_2.$$

On the other hand,

$$\|S_{\text{sep}}\|_{L_\infty} = \max_{t_0,t_1,t_2} |P_1(t_0) + P_2(t_1,t_2)| \leqslant \max_{t_0,t_1,t_2} \{|P_1(t_0)| + |P_2(t_1,t_2)|\} \leqslant M_1 + M_2.$$

Combining the two inequalities, we obtain the desired equality. (9) □

Corollary 3.2. *For mixed multiple Rademacher system polynomial as in Proposition* 3.1 *we have*

$$\|P_1\|_{L_\infty} \leqslant \|S_{sep}\|_{L_\infty},$$
$$\|P_2\|_{L_\infty} \leqslant \|S_{sep}\|_{L_\infty}.$$

From this we conclude that the following is true.

Theorem 3.3. *For all $n \in \mathbb{N}$ and $a_{ij} \in \mathbb{R}$, $1 \leqslant i, j \leqslant n$, we have the following two-sided estimates with equivalence constants independent of n, a_{ij}*

$$\mathsf{E}_{\theta_k,\theta_{ij}}\left[\Big\|\sum_{k=1}^{n}\theta_k b_k r_k + \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}a_{ij}r_i\otimes r_j\Big\|_{L_\infty}\right] \asymp \min_{\Theta_1,\Theta_2}\left[\Big\|\sum_{k=1}^{n}\theta_k b_k r_k + \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}a_{ij}r_i\otimes r_j\Big\|_{L_\infty}\right] \asymp \tag{10}$$
$$\asymp \sum_{k=1}^{n}|b_k| + \max\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}^2\right)^{1/2}, \sum_{j=1}^{n}\left(\sum_{i=1}^{n}a_{ij}^2\right)^{1/2}\right\},$$

where $\Theta_1 = (\theta_k)$ and $\Theta_2 = (\theta_{ij})$ are independent sequences of Rademacher signs.

As consequence, the mixed multiple Rademacher system has the RUC property.

Proof. We firstly prove the RUC property. If $\Theta_1 = (\theta_k) \in \{-1,1\}^n$ and $\Theta_2 = (\theta_{ij}) \in \{-1,1\}^n \times \times \{-1,1\}^n$ are independent random sign, then we put

$$\Theta_1 P_1 := \sum_{k=1}^{n}\theta_k b_k r_k, \quad \Theta_2 P_2 := \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}a_{ij}r_i\otimes r_j.$$

Now, because always $\mathsf{E}_{\Theta_1,\Theta_2}\|\Theta_1 P_1 + \Theta_2 P_2\|_{L_\infty} \geqslant \min_{\Theta_1,\Theta_2}\|\Theta_1 P_1 + \Theta_2 P_2\|_{L_\infty}$, it is enough for us to get an upper bound of expectation on signs:

$$\mathsf{E}_{\Theta_1,\Theta_2}\|\Theta_1 P_1 + \Theta_2 P_2\|_{L_\infty} = \mathsf{E}_{\theta_k,\theta_{ij}}\left[\|\sum_{k=1}^{n}\theta_k b_k r_k\|_{L_\infty} + \|\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}a_{ij}r_i\otimes r_j\|_{L_\infty}\right] =$$
$$= \mathsf{E}_{\theta_k}\|\sum_{k=1}^{n}\theta_k b_k r_k\|_{L_\infty} + \mathsf{E}_{\theta_{ij}}\|\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}a_{ij}r_i\otimes r_j\|_{L_\infty} \leqslant$$
$$\leqslant \min_{\theta_k}\|\sum_{k=1}^{n}\theta_k b_k r_k\|_{L_\infty} + C_{RUC}\min_{\theta_{ij}}\|\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}a_{ij}r_i\otimes r_j\|_{L_\infty} \leqslant$$
$$\leqslant C_{RUC}\min_{\Theta_1,\Theta_2}(\|\Theta_1 P_1\|_{L_\infty} + \|\Theta_2 P_2\|_{L_\infty}) \leqslant$$
$$\leqslant C_{RUC}\min_{\Theta_1,\Theta_2}\|\Theta_1 P_1 + \Theta_2 P_2\|_{L_\infty},$$

where the first equality comes by taking expectations on (Θ_1, Θ_2) from both sides of (9), the second equality by linearity of expectation and independence of (Θ_1, Θ_2), and third inequality from symmetric property of Rademacher system and from RUC property of second-degree homogeneous multiple Rademacher systems (by Theorem 2.2). In fact, the L_∞ norm of the first-degree Rademacher system with random signs is equal to sum of absolute values b_k, which corresponds to symmetric property of this system in L_∞. And then we use known properties of minima of functions and in the final inequality we use (9) again. Thus, the mixed multiple Rademacher system possesses the RUC property.

Now, to prove the second part of (10), we again use the Proposition 3.1 and the following simple fact:

$$\|\Theta_1 P_1\|_{L_\infty} = \sum_{k=1}^{n}|\theta_k b_k| = \|(b_k)\|_{l_1}. \tag{11}$$

Now, if we take $\tilde{b_k} = \theta_k b_k$ for fixed combinations of signs $\Theta_1 = (\theta_k)$, the same holds true. Then we unfix the signs and take expectation from both sides of the equality:

$$\mathsf{E}_{\Theta_1}\|\Theta_1 P_1\|_{L_\infty} = \mathsf{E}_{\theta_k}\sum_{k=1}^{n}|\theta_k b_k| = \|b_k\|_{l_1}. \tag{12}$$

For the second-degree homogeneous multiple Rademacher system polynomial $\Theta_2 P_2 = \sum_{i=1}^n \sum_{j=1}^n a_{ij} r_i \otimes r_j$, by Theorem 2.2, its average L_∞-norm is equivalent to the matrix norm:

$$\mathsf{E}_{\Theta_2} \|\Theta_2 P_2\|_{L_\infty} \asymp \max\left\{ \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 \right)^{1/2}, \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}^2 \right)^{1/2} \right\}. \tag{13}$$

Combining relations (12) and (13) with Proposition 3.1, we obtain the right-hand side of relation (10).

□

3.2. RUC property for mixed Rademacher chaos

Now we consider the mixed Rademacher chaos polynomials, as in (6). Let $S(t) = P_1(t) + P_2(t)$ be such a polynomial, with $P_1(t) = \sum_{k=1}^n b_k r_k(t)$ and $P_2(t) = \sum_{1 \leqslant i < j \leqslant n} a_{ij} r_i(t) r_j(t)$. In this case, the simple additivity of L_∞-norms observed in Proposition 3.1 no longer holds due to the mutual dependence between $P_1(t)$ and $P_2(t)$. However, a crucial relationship still provides control over the norm of its components by the norm of the total sum.

Proposition 3.4. *Let*

$$P_1(t) = \sum_{k=1}^n b_k r_k(t), \quad P_2(t) = \sum_{1 \leqslant i < j \leqslant n} a_{ij} r_i(t) r_j(t),$$

and $S(t) = P_1(t) + P_2(t)$ is a mixed Rademacher chaos polynomial of second degree as it defined in (6). *Then,*

$$\|P_1\|_{L_\infty} \leqslant \|S\|_{L_\infty}, \quad \|P_2\|_{L_\infty} \leqslant \|S\|_{L_\infty}. \tag{14}$$

As consequence,

$$\|S\|_{L_\infty} \leqslant \|P_1\|_{L_\infty} + \|P_2\|_{L_\infty} \leqslant 2\|S\|_{L_\infty}.$$

Proof. Let t^* be a point where $|P_1(t^*)| = \|P_1\|_{L_\infty}$. Without loss of generality, assume $P_1(t^*) = \|P_1\|_{L_\infty} \geqslant 0$. Consider another point t^{**} such that $r_k(t^{**}) = -r_k(t^*)$ for all $k = 1, \dots, n$. Then $P_1(t^{**}) = \sum b_k(-r_k(t^*)) = -P_1(t^*)$. However, $P_2(t^{**}) = \sum_{i<j} a_{ij}(-r_i(t^*))(-r_j(t^*)) = \sum_{i<j} a_{ij} r_i(t^*) r_j(t^*) = P_2(t^*)$. Thus, we have two values for $S(t)$: $S(t^*) = P_1(t^*) + P_2(t^*)$ and $S(t^{**}) = -P_1(t^*) + P_2(t^*)$. At least one of $P_1(t^*)$ or $-P_1(t^*)$ must have the same sign as $P_2(t^*)$.

If $P_1(t^*)$ and $P_2(t^*)$ have the same sign, then

$$|S(t^*)| = |P_1(t^*) + P_2(t^*)| = |P_1(t^*)| + |P_2(t^*)| \geqslant |P_1(t^*)| = \|P_1\|_{L_\infty}.$$

If $P_1(t^*)$ and $P_2(t^*)$ have opposite signs, then

$$|S(t^{**})| = |-P_1(t^*) + P_2(t^*)| = |P_1(t^*)| + |P_2(t^*)| \geqslant |P_1(t^*)| = \|P_1\|_{L_\infty}$$

In either case,

$$\|S\|_{L_\infty} = \max_t |S(t)| \geqslant \|P_1\|_{L_\infty}.$$

Thus, we get the first inequality in (14), and for the second inequality we can apply the similar argument.

Next, using inequalities (14) we get:

$$\|P_1\|_{L_\infty} + \|P_2\|_{L_\infty} \leqslant 2\|S\|_{L_\infty},$$

and by triangle inequality

$$\|S\|_{L_\infty} \leqslant \|P_1\|_{L_\infty} + \|P_2\|_{L_\infty}.$$

□

Corollary 3.5. *Let $X_1 = \overline{\mathrm{span}}\{r_k : k = 1, 2, 3, \dots\}$ be the closed subspace of first-degree homogeneous Rademacher chaos in $L_\infty([0,1])$, $X_2 = \overline{\mathrm{span}}\{r_i r_j : i < j, i, j = 1, 2, 3, \dots\}$ be the closed subspace of second-degree homogeneous Rademacher chaos in $L_\infty([0,1])$, and $X_{1,2} = \overline{\mathrm{span}}\{r_i r_j, r_k : i < j, i, j, k = 1, 2, 3, \dots\}$ be the closed subspace of second-degree mixed Rademacher chaos in $L_\infty([0,1])$. Then there is an isomorphism of Banach spaces:*

$$X_{1,2} \cong X_1 \oplus X_2,$$

where

$$X_1 \oplus X_2 := \{x = x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}, \quad \|x_1 + x_2\|_{X_1 \oplus X_2} := \|x_1\|_{X_1} + \|x_2\|_{X_2}.$$

Thus X_1 and X_2 are complemented subspaces of $X_{1,2}$.

Proof. Mixed chaos is a basic sequence in lexicographic order in the space L_∞ [32, Theorem 2]. Therefore, the space $X_{1,2}$ consists of those elements $x_{1,2} \in L_\infty$ which can be represented in the form

$$x_{1,2}(t) = b_1 r_1(t) + b_2 r_2(t) + b_{1,2} r_1(t) r_2(t) + b_3 r_3(t) + b_{1,3} r_1(t) r_3(t) + b_{2,3} r_2(t) r_3(t) + \ldots, \tag{15}$$

where the series converges in the L_∞-norm. Similarly, the spaces X_1 and X_2 consist, respectively, of elements which are represented as sums of the form

$$x_1 = a_1 r_1(t) + a_2 r_2(t) + a_3 r_3(t) + \ldots \quad \text{and} \quad x_2 = a_{1,2} r_1(t) r_2(t) + a_{1,3} r_1(t) r_3(t) + a_{2,3} r_2(t) r_3(t) + \ldots$$

Let us consider two arbitrary elements $x_1 \in X_1$ and $x_2 \in X_2$. It is easy to see that from the convergence of the series for x_1 and x_2 follows the convergence of the series

$$x_1 + x_2 = a_1 r_1(t) + a_2 r_2(t) + a_{1,2} r_1(t) r_2(t) + a_3 r_3(t) + a_{1,3} r_1(t) r_3(t) + a_{2,3} r_2(t) r_3(t) + \ldots,$$

therefore $X_1 + X_2 \subset X_{1,2}$. Then we note that convergence in L_∞ implies convergence in L_2, in which the spaces X_1 and X_2 are orthogonal. Therefore $X_1 \cap X_2 = \{0\}$, and the space $X_1 \oplus X_2$ is well defined. Moreover,

$$\|x_1 + x_2\|_{X_{1,2}} = \|x_1 + x_2\|_{L_\infty} \leqslant \|x_1\|_{L_\infty} + \|x_2\|_{L_\infty} = \|x_1 + x_2\|_{X_1 \oplus X_2}.$$

Now let $x_{1,2} \in X_{1,2}$ and $S_{1,2}^{(n)}$ be the partial sum of the corresponding series (15). Then $S_{1,2}^{(n)} = S_1^{(n_1)} + S_2^{(n_2)}$, where $S_1^{(n_1)}$ is some finite sum according to system $\{r_k\}$, and $S_2^{(n_2)}$ is a finite sum according to system $\{r_i r_j\}$. As n increases, new terms will be added to the sums $S_1^{(n_1)}$ and $S_2^{(n_2)}$ in a certain order determined by the lexicographic numbering of the combined system of $\{r_k\}$ and $\{r_i r_j\}$. The sequence of sums $S_1^{n_1}$ will form the series

$$b_1 r_1(t) + b_2 r_2(t) + b_3 r_3(t) + \ldots,$$

and, similarly, the sequence of sums $S_2^{(n_2)}$ will form the series

$$b_{1,2} r_1(t) r_2(t) + b_{1,3} r_1(t) r_3(t) + b_{2,3} r_2(t) r_3(t) + \ldots.$$

Both of these series will converge. This follows from the convergence of the series for $x_{1,2}$ and inequalities

$$\left\|S_1^{(m_1)} - S_1^{(n_1)}\right\|_{L_\infty} \leqslant \left\|S_{1,2}^{(m)} - S_{1,2}^{(n)}\right\|_{L_\infty} \quad \text{and} \quad \left\|S_2^{(m_2)} - S_2^{(n_2)}\right\|_{L_\infty} \leqslant \left\|S_{1,2}^{(m)} - S_{1,2}^{(n)}\right\|_{L_\infty}, \quad n < m,$$

which are valid by virtue of Proposition 3.4. Hence $x_{1,2} = x_1 + x_2$, where $x_i \in X_i$, and $X_{1,2} \subset X_1 \oplus X_2$. By virtue of the already established continuous embedding $X_1 \oplus X_2 \subset X_{1,2}$ and Banach's inverse operator theorem, embedding $X_{1,2} \subset X_1 \oplus X_2$ is also continuous. Moreover, passing to the limit in inequalities

$$\left\|S_1^{(n_1)}\right\|_{L_\infty} \leqslant \left\|S_{1,2}^{(n)}\right\|_{L_\infty} \quad \text{and} \quad \left\|S_2^{(n_2)}\right\|_{L_\infty} \leqslant \left\|S_{1,2}^{(n)}\right\|_{L_\infty}, \quad n < m,$$

which are valid according to Proposition 3.4, we obtain

$$\|x_{1,2}\|_{X_{1,2}} \leqslant \|x_{1,2}\|_{X_1 \oplus X_2} \leqslant 2\|x_{1,2}\|_{X_{1,2}}.$$

□

Now we prove the RUC property for mixed Rademacher chaos. We proceed similarly to Theorem 3.3.

Theorem 3.6. *For all $n \in \mathbb{N}$ and $a_{ij} \in \mathbb{R}$, $1 \leqslant i < j \leqslant n$, we have the following two-sided estimates with equivalence constants independent of n, a_{ij}*

$$\mathsf{E}_{\Theta_1,\Theta_2} \left\| \sum_{k=1}^n \theta_k b_k r_k + \sum_{1 \leqslant i < j \leqslant n} \theta_{ij} a_{ij} r_i r_j \right\|_{L_\infty} \asymp \min_{\Theta_1,\Theta_2} \left\| \sum_{k=1}^n \theta_k b_k r_k + \sum_{1 \leqslant i < j \leqslant n} \theta_{ij} a_{ij} r_i r_j \right\|_{L_\infty} \asymp \tag{16}$$

$$\asymp \sum_{k=1}^n |b_k| + \max \left\{ \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^n a_{ij}^2 \right)^{1/2}, \sum_{j=2}^n \left(\sum_{i=1}^{j-1} a_{ij}^2 \right)^{1/2} \right\},$$

where $\Theta_1 = (\theta_k)$ and $\Theta_2 = (\theta_{ij})$ are independent sequences of Rademacher signs.

As consequence, the mixed Rademacher chaos has the RUC property.

Proof. Let us denote

$$\Theta_1 P_1 := \sum_{k=1}^{n} \theta_k b_k r_k, \quad \Theta_2 P_2 := \sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i(t) r_j(t),$$

as in the proof of Theorem 3.3, we obtain

$$\begin{aligned}
\mathsf{E}_{\Theta_1,\Theta_2} \|(\Theta_1 P_1) + (\Theta_2 P_2)\|_{L_\infty} &\leqslant \mathsf{E}_{\theta_k,\theta_{ij}} \left[\|\sum_{k=1}^{n} \theta_k b_k r_k\|_{L_\infty} + \|\sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i r_j\|_{L_\infty} \right] = \\
&= \mathsf{E}_{\theta_k} \|\sum_{k=1}^{n} \theta_k b_k r_k\|_{L_\infty} + \mathsf{E}_{\theta_{ij}} \|\sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i r_j\|_{L_\infty} \leqslant \\
&\leqslant \min_{\theta_k} \|\sum_{k=1}^{n} \theta_k b_k r_k\|_{L_\infty} + C_{RUC} \min_{\theta_{ij}} \|\sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i r_j\|_{L_\infty} \leqslant \\
&\leqslant C_{RUC} \min_{\Theta_1,\Theta_2} (\|\Theta_1 P_1\|_{L_\infty} + \|\Theta_2 P_2\|_{L_\infty}) \leqslant \\
&\leqslant 2C_{RUC} \min_{\Theta_1,\Theta_2} \|\Theta_1 P_1 + \Theta_2 P_2\|_{L_\infty},
\end{aligned}$$

where final inequality comes from Proposition 3.4. From this we obtain the RUC-property for the mixed Rademacher chaos.

To prove the second equivalence in (16), we again use Proposition 3.4. Thus we get

$$\begin{aligned}
\mathsf{E}_{\Theta_1,\Theta_2} \|\Theta_1 P_1 + \Theta_2 P_2\|_{L_\infty} &\asymp \mathsf{E}_{\theta_k} \|\sum_{k=1}^{n} \theta_k b_k r_k\|_{L_\infty} + \mathsf{E}_{\theta_{ij}} \|\sum_{1\leqslant i<j\leqslant n} \theta_{ij} a_{ij} r_i r_j\|_{L_\infty} \asymp \\
&\asymp \sum_{k=1}^{n} |b_k| + \max \left\{ \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} a_{ij}^2 \right)^{1/2}, \sum_{j=2}^{n} \left(\sum_{i=1}^{j-1} a_{ij}^2 \right)^{1/2} \right\},
\end{aligned}$$

where we used relations (11) and (1). □

4. RUC property for multiple random system and chaos of symmetric bounded random variables

In this section we extend results from Section 3 obtained for second-degree mixed multiple Rademacher system and mixed Rademacher chaos to broader class of d-th degree mixed multiple random system and mixed chaos of symmetric bounded (a. e.) rvs.

4.1. RUC property for homogeneous multiple random system and homogeneous chaos

First we will establish equality between L_∞-norm of homogeneous multiple random system polinomial and the L_∞-norm of homogeneous multiple Rademacher system polynomial of degree d. As before, we will denote by $X^{(1)},\dots,X^{(d)}$ independent copies of the sequence $X = (X_k)$.

Theorem 4.1. *Let $\{X_{j_1}^{(1)}\dots X_{j_d}^{(d)}\}$ be a d-homogeneous multiple random system formed by the sequence $X = (X_k)$ of independent symmetric bounded random variables, and $\|X_k\|_{L_\infty} = C_k > 0$. Let*

$$P(X^{(1)},\dots,X^{(d)}) = \sum_{J\in\mathbb{N}_n^d} a_J X_{j_1}^{(1)}\dots X_{j_d}^{(d)}$$

is a polynomial by this system. Then,

$$\|P(X^{(1)},\dots,X^{(d)})\|_{L_\infty} = \left\| \sum_{J\in\mathbb{N}_n^d} \left(a_J \prod_{l\in J} C_l \right) r_J^{\otimes} \right\|_{L_\infty},$$

where $r_J^{\otimes}$ denotes the elements of the d-th degree multiple Rademacher system.

Proof. We remind that we work on a probability space (Ω, A, P) with $\Omega = [0,1]$ an $\mathsf{P} = \mu$, where μ is a standard Lebesgue measure. Let us consider an auxiliary multilinear form

$$P_h(x^{(1)},\dots,x^{(d)}) := \sum_{J\in\mathbb{N}_n^d} a_J x_{j_1}^{(1)}\dots x_{j_d}^{(d)},$$

which depends on variables $x^{(1)} = (x_1^{(1)},\dots,x_n^{(1)})$, ..., $x^{(d)} = (x_1^{(d)},\dots,x_n^{(d)})$, and each $x^{(k)}$, $k\in[d]$, changes on a Cartesian product $\prod_{i=1}^n[-C_i,C_i]$. We show that the L_∞-norm of the d-homogeneous multiple random system polynomial formed by independent symmetric bounded random variables X_k coincides with the L_∞-norm of this multilinear form.

To show that $\|P_h\|_{L_\infty} = \|\sum_{J\in\mathbb{N}_n^d} a_J X_{j_1}^{(1)}\dots X_{j_d}^{(d)}\|_{L_\infty}$, we proof two inequalities, which will give us the desired equality when combined.

Firstly, we note that the inequality $\|P_h\|_{L_\infty} \geqslant \|\sum_{J\in\mathbb{N}_n^d} a_J X_{j_1}^{(1)}\dots X_{j_d}^{(d)}\|_{L_\infty}$ holds true, because the co-domain of random variable $P(X^{(1)},\dots,X^{(d)})$ is included in the co-domain of $P_h(x^{(1)},\dots,x^{(d)})$ almost surely. Note that this holds for arbitrary independent symmetric bounded rvs (X_i) with norm $\|X_i\|_{L_\infty} = C_i > 0$.

Now we prove the inverse inequality. By multilinearity of the form P_h we have that

$$M := \|P_h\|_{L_\infty} = \max_{|x_{j_k}^{(k)}|=C_{j_k}} \left|\sum_{J\in\mathbb{N}_n^d} a_J x_{j_1}^{(1)}\dots x_{j_d}^{(d)}\right| = \sum_{J\in\mathbb{N}_n^d} a_J C^*_{1,j_1}\dots C^*_{d,j_d},$$

where we note C^*_{i,j_i} equal to C_{j_i} or $-C_{j_i}$, depending on where the maximum is attained.

We consider the following family of sets:

$$\Omega_{i,j_i,\epsilon} := \left\{\omega\in\Omega \mid X_{j_i}^{(i)}(\omega)\in\Delta_{i,j_i,\epsilon}\right\},\quad i\in[d],\, j_i\in[n],$$

where by $\Delta_{i,j_i,\epsilon}$ we denote either the interval $[C^*_{i,j_i}-\epsilon, C^*_{i,j_i}]$ or the interval $[C^*_{i,j_i}, C^*_{i,j_i}+\epsilon]$, again, depending on the sign of C^*_{i,j_i}. By definition of essential supremum, we have that $\mathsf{P}(\Omega_{i,j_i,\epsilon}) > 0$.

Let us consider the set

$$\Omega_\epsilon = \bigcap_{i=1}^d \bigcap_{j_i=1}^n \Omega_{i,j_i,\epsilon}.$$

By independence rvs from the system $\{(X_k^{(i)})_k\}_i$, we have that

$$\mathsf{P}(\Omega_\epsilon) = \prod_{i=1}^d \prod_{j_i=1}^n \mathsf{P}(\Omega_{i,j_i,\epsilon}),$$

so that $\mathsf{P}(\Omega_\epsilon) > 0$ as the product of positive measures. By definition of Ω_ϵ we have inclusions

$$\left\{P(X^{(1)},\dots,X^{(d)}) \mid \omega\in\Omega_\epsilon\right\} \subset \left\{P_h \mid x_{j_i}^{(i)}\in\Delta_{i,j_i,\epsilon}\right\} = [M-\delta(\epsilon), M],$$

with some $\delta(\epsilon)$. Moreover, $\delta(\epsilon)\to 0$ with $\epsilon\to 0$ by continuity of P_h.

From here we have that

$$|P(X^{(1)},\dots,X^{(d)}| \geqslant M-\delta(\epsilon)$$

on a set with positive measure. From this we get

$$\|P(X^{(1)},\dots,X^{(d)}\|_{L_\infty} \geqslant \lim_{\epsilon\to 0}\{M-\delta(\epsilon)\} = M.$$

Thus,

$$\begin{aligned}\|P(X^{(1)},\dots,X^{(d)})\|_{L_\infty} &= \sup_{|x_{j_k}^{(k)}|=C_{j_k}} \left|\sum_J a_J x_{j_1}^{(1)}\dots x_{j_d}^{(d)}\right| = \\ &= \max_{\epsilon_k^{(m)}\in\{\pm1\}} \left|\sum_J a_J (C_{j_1}\epsilon_{j_1}^{(1)})\dots(C_{j_d}\epsilon_{j_d}^{(d)})\right| = \\ &= \max_{\epsilon_k^{(m)}\in\{\pm1\}} \left|\sum_J \left(a_J \prod_{l\in J} C_l\right)\epsilon_{j_1}^{(1)}\dots\epsilon_{j_d}^{(d)}\right|.\end{aligned}$$

The last expression is precisely the L_∞-norm of the d-th degree multiple Rademacher system with modified coefficients $\tilde{a}_J = a_J \prod_{l\in J} C_l$. Therefore,

$$\|P(X^{(1)},\dots,X^{(d)})\|_{L_\infty} = \left\| \sum_{J\in\mathbb{N}_n^d} \left(a_J \prod_{l\in J} C_l \right) r_J^{\otimes} \right\|_{L_\infty}.$$

□

The following statement follows from Theorem 4.1, Theorem 2.2 and averaging over cyclic permutations of the index k in inequality (3).

Theorem 4.2. *The d-degree homogeneous multiple random system formed by the sequence $X = (X_k)$ of independent symmetric bounded rvs has the RUC property. Moreover, for all $n \in \mathbb{N}$ and $a_J \in \mathbb{R}^d, J \in \mathbb{N}_n^d$ the following inequalities hold:*

$$\left\| \sum_{J=(j_1,j_2,\dots,j_d)\in\mathbb{N}_n^d} a_J X_{j_1}^{(1)} X_{j_2}^{(2)} \dots X_{j_d}^{(d)} \right\|_{L_\infty([0,1]^d)} \geqslant 2^{\frac{1-d}{2}} \max_{k\in[d]} \sum_{l=1}^n \left(\sum_{J\in\mathbb{N}_n^d(k,l)} \tilde{a}_J^2 \right)^{1/2},$$

and

$$\mathsf{E}_\theta \left\| \sum_{J=(j_1,j_2,\dots,j_d)\in\mathbb{N}_n^d} a_J \theta_J X_{j_1}^{(1)} X_{j_2}^{(2)} \dots X_{j_d}^{(d)} \right\|_{L_\infty([0,1]^d)} \leqslant \frac{2^d-1}{d} \sum_{k=1}^d \sum_{l=1}^n \left(\sum_{J\in\mathbb{N}_n^d(k,l)} \tilde{a}_J^2 \right)^{1/2},$$

where $(\theta_J)_{J\in\mathbb{N}_n^d} = \pm 1$ is a system of independent symmetric random signs, $\tilde{a}_J = a_J \prod_{l\in J} C_l$, $C_l = \|X_l\|_{L_\infty} > 0$.

Using the last fact for a homogeneous multiple random system and Lemma 2.1, we can establish the RUC property for homogeneous chaos.

Theorem 4.3. *The d-degree homogeneous chaos formed by the sequence $X = (X_k)$ of independent symmetric bounded rvs, $\|X_k\|_{L_\infty} = C_k > 0$, has the RUC property. Moreover, the following relations are valid with constants depending only on d*

$$\mathsf{E}_\theta \left\| \sum_{J\in\Delta_n^d} \theta_J a_J X_J \right\|_{L_\infty} \asymp \min_\theta \left\| \sum_{J\in\Delta_n^d} \theta_J a_J X_J \right\|_{L_\infty} \asymp \sum_{j=1}^n \left(\sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ \exists k\in[d]:\ j_k=j}} \tilde{a}_J^2 \right)^{1/2}, \tag{17}$$

where $(\theta_J)_{J\in\Delta_n^d} = \pm 1$ is a system of independent symmetric random signs, $\tilde{a}_J = a_J \prod_{l\in J} C_l$, $X_J = X_{j_1} X_{j_2} \dots X_{j_d}$.

Proof. For the proof we will use Theorem 4.2 and Lemma 2.1. Let b_J for $J \in \mathbb{N}_n^d$ be defined as following:

$$b_{j_1,\dots,j_d} = \frac{1}{d!} a_{j_{\sigma_1}} \dots a_{j_{\sigma_d}}, \text{if all } j_i \text{ are pairwise different,}$$

where σ is permutation of $[d]$ such that $j_{\sigma_1} < j_{\sigma_2} < \dots < j_{\sigma_d}$, and $b_{j_1,\dots,j_d} = 0$ if there exists a pair (j_{i_1}, j_{i_2}) such that $j_{i_1} = j_{i_2}$ for $i_1 \neq i_2$. These coefficients satisfy conditions of Lemma 2.1, and

$$\sum_{i_1=1}^n \dots \sum_{i_d=1}^n b_{j_1,\dots,j_d} X_{j_1} \dots X_{j_d} = \sum_{1<i_1<i_2<\dots<i_d} a_{j_1,\dots,j_d} X_{j_1} \dots X_{j_d}.$$

Let $\tilde{b}_J = b_J \prod_{l\in J} C_l$. By Lemma 2.1 and Theorem 4.2, we get

$$\begin{aligned}
\left\| \sum_{1<i_1<i_2<\dots<i_d} a_{j_1,\dots,j_d} X_{j_1} \dots X_{j_d} \right\|_{L_\infty} &\geqslant c_d \left\| \sum_{i_1=1}^n \dots \sum_{i_d=1}^n b_{j_1,\dots,j_d} X_{j_1}^{(1)} \dots X_{j_d}^{(d)} \right\|_{L_\infty} \geqslant \\
&\geqslant c_d 2^{\frac{1-d}{2}} \max_{k\in[d]} \sum_{j=1}^n \left(\sum_{J\in\mathbb{N}_n^d(k,j)} \tilde{b}_J^2 \right)^{1/2} = \\
&= c_d 2^{\frac{1-d}{2}} \sum_{j=1}^n \left(\sum_{J\in\mathbb{N}_n^d(1,j)} \tilde{b}_J^2 \right)^{1/2},
\end{aligned}$$

where the last equality follows from the symmetry of the system of coefficients $\{\tilde{b}_J\}_{J\in\mathbb{N}_n^d}$. Further, due to the same symmetry,

$$\sum_{J\in\mathbb{N}_n^d(1,j)} \tilde{b}_J^2 = \sum_{J\in\mathbb{N}_n^d:\, j_1=j} \tilde{b}_J^2 = \frac{1}{d} \sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\mathbb{N}_n^d:\\ \exists k\in[d]\,:\, j_k=j}} \tilde{b}_J^2 = \frac{1}{d} \sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ \exists k\in[d]\,:\, j_k=j}} \tilde{a}_J^2.$$

Hence

$$\Big\| \sum_{1<i_1<i_2<\dots<i_d} a_{j_1,\dots,j_d} X_{j_1}\dots X_{j_d} \Big\|_{L_\infty} \geqslant \frac{c_d}{\sqrt{d}2^{\frac{d-1}{2}}} \sum_{j=1}^n \Bigg(\sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ \exists k\in[d]\,:\, j_k=j}} \tilde{a}_J^2 \Bigg)^{1/2}.$$

To obtain the upper estimate for the expectation E_θ, we cannot directly use the decoupling method. The difficulty arises because moving to separated chaos requires averaging over signs Θ_J by all multi-indices $J\in\mathbb{N}_n^d$, not just ascending ones from Δ_n^d. To overcome this, we use the reasoning following Lemma 2.1 and establish the inequality:

$$\Big\| \sum_{J\in\Delta_n^d} \theta_J a_J X_J \Big\|_{L_\infty} \leqslant \Big\| \sum_{J\in\Delta_n^d} \theta_J a_J X_{j_1}^{(1)} X_{j_2}^{(2)} \dots X_{j_d}^{(d)} \Big\|_{L_\infty}$$

for each set of signs $\{\theta_J\}_{J\in\Delta_n^d}$. Therefore, by Theorem 4.2, where we put $\tilde{a}_J = 0$ for $J\notin\Delta_n^d$,

$$\begin{aligned}
\mathsf{E}_\theta \Big\| \sum_{J\in\Delta_n^d} \theta_J a_J X_J \Big\|_{L_\infty} &\leqslant \mathsf{E}_\theta \Big\| \sum_{J\in\Delta_n^d} \theta_J a_J X_{j_1}^{(1)} X_{j_2}^{(2)} \dots X_{j_d}^{(d)} \Big\|_{L_\infty} \leqslant \\
&\leqslant \frac{2^d-1}{d} \sum_{k=1}^d \sum_{j=1}^n \Bigg(\sum_{J\in\mathbb{N}_n^d(k,j)} \tilde{a}_J^2 \Bigg)^{1/2} = \\
&= \frac{2^d-1}{d} \sum_{j=1}^n \sum_{k=1}^d \Bigg(\sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ j_k=j}} \tilde{a}_J^2 \Bigg)^{1/2} \leqslant \\
&\leqslant \frac{2^d-1}{\sqrt{d}} \sum_{j=1}^n \Bigg(\sum_{k=1}^d \sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ j_k=j}} \tilde{a}_J^2 \Bigg)^{1/2} = \\
&= \frac{2^d-1}{\sqrt{d}} \sum_{j=1}^n \Bigg(\sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ \exists k\in[d]\,:\, j_k=j}} \tilde{a}_J^2 \Bigg)^{1/2}. \qquad \square
\end{aligned}$$

Corollary 4.4. *Let $d\geqslant 2$. The d-degree homogeneous chaos is not an unconditional system. This means that there is no constant C such that for all $n\in\mathbb{N}$, $a_J\in\mathbb{R}^d$ and $\theta_J=\pm 1$, $J\in\Delta_n^d$, the following inequality holds:*

$$\Big\| \sum_{J\in\Delta_n^d} a_J X_J \Big\|_{L_\infty} \leqslant C \Big\| \sum_{J\in\Delta_n^d} \theta_J a_J X_J \Big\|_{L_\infty}.$$

Similary, the d-degree homogeneous multiple random system is not an unconditional system.

Proof. Without loss of generality, we can assume that $\|X_k\|_{L^\infty}=1$, $k=1,2,\dots$. Let us take $a_J=1$. Then

$$\Big\| \sum_{J\in\Delta_n^d} a_j X_J \Big\|_{L_\infty} = \Big\| \sum_{J\in\Delta_n^d} X_J \Big\|_{L_\infty} = C_n^d,$$

where $C_n^d = \frac{n!}{d!(n-d)!}$. This follows from the fact that for any $\epsilon>0$

$$\mathsf{P}\Bigg(\prod_{k=1}^n \{X_k > 1-\epsilon\} \Bigg) > 0,$$

and from the continuity of the polynomial form

$$\sum_{J\in\Delta_n^d} x_J, \quad x_J = x_{j_1}x_{j_2}\dots x_{j_d}.$$

On the other hand, according to (17),

$$\min_{\theta}\left\|\sum_{J\in\Delta_n^d}\theta_J X_J\right\|_{L_\infty} \leqslant C\sum_{j=1}^{n}\left(\sum_{\substack{J=(j_1,j_2,\dots,j_d)\in\Delta_n^d:\\ \exists k\in[d]:\, j_k=j}} 1^2\right)^{1/2} = Cn\sqrt{C_{n-1}^{d-1}},$$

with some constant C that does not depend on n. Since

$$C_n^d \asymp n^d, \quad cn\sqrt{C_{n-1}^{d-1}} \asymp n^{\frac{d+1}{2}},$$

where the equivalence constants depend only on d, the unconditional inequality cannot be satisfied, as for $d \geqslant 2$ we get $d > \dfrac{d+1}{2}$. □

An analogous fact can be established for the d-degree homogeneous multiple system by using a similar argument.

4.2. RUC property for mixed multiple random system and mixed chaos

We will first establish a key property for the L_∞-norm of mixed multiple system polynomial generated by symmetric bounded random variables, analogous to Proposition 3.1 for Rademacher variables.

Proposition 4.5. *Let*

$$S_{sep}(X^{(1)},X^{(2)},\dots,X^{(\frac{d(d+1)}{2})}) = P_1(X^{(1)})+P_2(X^{(2)},X^{(3)})+\dots+P_d(X^{(1+\frac{d(d-1)}{2})},X^{(2+\frac{d(d-1)}{2})},\dots,X^{(\frac{d(d+1)}{2})})$$

be a d-degree mixed multiple system polynomial, where $X^{(1)},X^{(2)},\dots X^{(d(d+1)/2)}$ are independent copies of a sequence $X=(X_k)$ of independent symmetric bounded variables. Then,

$$\|S_{sep}\|_{L_\infty} = \|P_1\|_{L_\infty}+\|P_2\|_{L_\infty}+\dots+\|P_d\|_{L_\infty}.$$

The assertion follows easily from the mutual independence of the terms $P_1,P_2,\dots$, and the following simple property.

Lemma 4.6. *Let ξ and η be independent symmetric bounded random variables. Then*

$$\|\xi+\eta\|_{L_\infty} = \|\xi\|_{L_\infty}+\|\eta\|_{L_\infty}.$$

Proof. Let

$$A = \|\xi\|_{L_\infty}, \quad B = \|\eta\|_{L_\infty}.$$

Due to the symmetry of random variables ξ and η, for any $\varepsilon>0$, events

$$\Omega_{\xi,\varepsilon} := \{\xi\in[A-\varepsilon,A]\} \quad \text{and} \quad \Omega_{\eta,\varepsilon} := \{\eta\in[B-\varepsilon,B]\}$$

have positive measure. From the independence of ξ and η it follows that

$$\mathsf{P}(\Omega_{\xi,\varepsilon}\cap\Omega_{\eta,\varepsilon})>0.$$

Moreover,

$$(\xi+\eta)(\Omega_{\xi,\varepsilon}\cap\Omega_{\eta,\varepsilon}) \subset [A+B-2\varepsilon, A+B].$$

Hence

$$\|\xi+\eta\|_{L_\infty} \geqslant \lim_{\varepsilon\to 0}(A+B-2\varepsilon) = A+B.$$

The opposite inequality coincides with the triangle inequality. □

Now, analogously to the proof of Theorem 3.3 from Proposition 4.5 and Theorem 4.2 we conclude that the following is true.

Theorem 4.7. *The mixed multiple random system from the sequence (X_k) of independent symmetric bounded rvs has the RUC property. Moreover, let*

$$S_{sep}(X^{(1)},X^{(2)},\dots,X^{(\frac{d(d+1)}{2})},\Theta) =$$

$$= \Theta_1 P_1(X^{(1)}) + \Theta_2 P_2(X^{(2)}, X^{(3)}) + \ldots + \Theta_d P_d(X^{(1+\frac{d(d-1)}{2})}, X^{(2+\frac{d(d-1)}{2})}, \ldots, X^{(\frac{d(d+1)}{2})}),$$

where

$$\Theta_m P_m := \sum_{(j_1, j_2, \ldots, j_m) \in \mathbb{N}_n^m} \theta_{j_1 j_2 \ldots j_m} a_{j_1 j_2 \ldots j_m} X_{j_1}^{(1+\frac{m(m-1)}{2})} X_{j_2}^{(2+\frac{m(m-1)}{2})} \ldots X_{j_m}^{(\frac{m(m+1)}{2})}.$$

Then

$$\mathsf{E}_\Theta \|S_{sep}(X^{(1)}, X^{(2)}, \ldots, X^{(\frac{d(d+1)}{2})}, \Theta)\|_{L_\infty} \asymp \min_\Theta \|S_{sep}(X^{(1)}, X^{(2)}, \ldots, X^{(\frac{d(d+1)}{2})}, \Theta)\|_{L_\infty} \asymp$$

$$\asymp \sum_{m=1}^{d} \max_{k \in [m]} \sum_{l=1}^{n} \left(\sum_{J \in \mathbb{N}_n^m(k,l)} \tilde{a}_J^2 \right)^{1/2},$$

where Θ is the system of independent signs $\theta = \pm 1$, $\tilde{a}_J = a_J \prod_{l \in J} C_l$, $C_l = \|X_l\|_{L_\infty} > 0$, and a constants in the designated equivalences do not depend on n and real numbers $\{a_{j_1 j_2 \ldots j_m}\}_{m=1}^d$ (but depend on d).

In the special case of mixed chaos of the second degree we obtain the following statement.

Corollary 4.8. *The mixed multiple random system $\{X_k^{(1)}, X_i^{(2)} X_j^{(3)}\}$ from independent symmetric bounded rvs has the RUC property, and we have the following inequalities:*

$$\mathsf{E}_{\theta_k, \theta_{ij}} \left\| \sum_{k=1}^{n} \theta_k b_k X_k^{(1)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} a_{ij} X_i^{(2)} X_j^{(3)} \right\|_{L_\infty} \asymp \min_{\theta_k, \theta_{ij}} \left\| \sum_{k=1}^{n} \theta_k b_k X_k^{(1)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} a_{ij} X_i^{(2)} X_j^{(3)} \right\|_{L_\infty} \asymp$$

$$\asymp \sum_{k=1}^{n} |\tilde{b}_k| + \max \left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{ij}^2 \right)^{1/2}, \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \tilde{a}_{ij}^2 \right)^{1/2} \right\},$$

where $\tilde{b}_k = C_k b_k, \tilde{a}_{ij} = C_i C_j a_{ij}$, $C_k = \|X_k\|_{L_\infty} > 0$.

To obtain an analogue of Theorem 4.7 for mixed chaos, we first note the following statement, similar to Proposition 4.5.

Proposition 4.9. *Let*

$$S(X) = P_1(X) + P_2(X) + \ldots + P_d(X)$$

be a d-degree mixed chaos polynomial, decomposed into the sum of its homogeneous components $P_m(X)$, where $X = (X_k)$ is a sequence of independent symmetric bounded variables. Then,

$$\|S\|_{L_\infty} \asymp \|P_1\|_{L_\infty} + \|P_2\|_{L_\infty} + \ldots + \|P_d\|_{L_\infty}.$$

Proof. From Corollary 2.5 we obtain

$$\|P_1\|_{L_\infty} + \|P_2\|_{L_\infty} + \ldots + \|P_d\|_{L_\infty} \leqslant d K_d \|S\|_{L_\infty}.$$

The opposite estimate is obtained from the triangle inequality. □

From Proposition 4.9 and Theorem 4.3 we get

Theorem 4.10. *The mixed chaos from the sequence (X_k) of independent symmetric bounded rvs has the RUC property. Morever, let*

$$S(X, \Theta) = \Theta_1 P_1(X) + \Theta_2 P_2(X) + \ldots + \Theta_d P_d(X),$$

where

$$\Theta_m P_m := \sum_{J \in \Delta_n^m} \theta_J a_J X_J, \quad X_J = X_{j_1} X_{j_2} \ldots X_{j_m}.$$

Then

$$\mathsf{E}_\Theta \|S(X, \Theta)\|_{L_\infty} \asymp \min_\Theta \|S(X, \Theta)\|_{L_\infty} \asymp$$

$$\asymp \sum_{m=1}^{d} \sum_{j=1}^{n} \left(\sum_{\substack{J = (j_1, j_2, \ldots, j_m) \in \Delta_n^m : \\ \exists k \in [d] : j_k = j}} \tilde{a}_J^2 \right)^{1/2},$$

where Θ *is the system of independent signs* $\theta = \pm 1$*,* $\tilde{a}_J := a_J \prod_{l \in J} C_l$*,* $C_l = \|X_l\|_{L_\infty} > 0$*, and a constants in the designated equivalences do not depend on n and real numbers* $\{\{a_J\}_{J \in \Delta_n^m}\}_{m=1}^d$ *(but depend on d).*

Corollary 4.11. *Let* (X_k) *is a sequence of independent symmetric bounded rvs. We have the following two-sided estimates*

$$\mathsf{E}_{\theta_k,\theta_{i,j}} \left\| \sum_{k=1}^n \theta_k b_k X_k + \sum_{1 \leqslant i < j \leqslant n} \theta_{ij} a_{ij} X_i X_j \right\|_{L_\infty} \asymp \min_{\theta_k,\theta_{i,j}} \left\| \sum_{k=1}^n \theta_k b_k X_k + \sum_{1 \leqslant i < j \leqslant n} \theta_{ij} a_{ij} X_i X_j \right\|_{L_\infty} \asymp$$

$$\asymp \sum_{k=1}^n |\tilde{b}_k| + \max \left\{ \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^n \tilde{a}_{ij}^2 \right)^{1/2}, \sum_{j=2}^n \left(\sum_{i=1}^{j-1} \tilde{a}_{ij}^2 \right)^{1/2} \right\},$$

where $(\theta_k = \pm 1, \theta_{i,j} = \pm 1)$ *is the system of independent signs,* $\tilde{b}_k = C_k b_k, \tilde{a}_{ij} = C_i C_j a_{ij}$*,* $C_k = \|X_k\|_{L_\infty} > 0$*.*

Summary

This paper investigates systems composed of products of independent random variables and their properties related to the additive decomposition of other random variables over such systems. These representations are closely related to the well-known Polynomial Chaos Expansion (PCE, see [33]) and are a special case of the generalized polynomial chaos (see [34–36]), which has numerous applications in mathematical modeling and machine learning. We show that for symmetric bounded random variables, these product systems, while failing to be unconditional convergence systems in the space L_∞ of bounded random variables, nonetheless possess the closely related property of Random Unconditional Convergence (RUC). Following the principle of moving from particular and simple cases to more general and complex ones, we sequentially examine the cases of Rademacher random variables (in Section 3) and arbitrary symmetric bounded random variables (in Section 4). We consider two variants of these product systems. In the first, simpler variant, each product involves factors from different independent copies of the generating sequence of random variables (Theorems 3.3, 4.2, and 4.7). In the second variant, each product consists of factors from one common sequence (Theorems 3.6, 4.3, and 4.10), which creates a more complex dependence structure between the elements of the constructed system. We also made a transition from homogeneous systems, where all products consist of the same number of factors (Theorems 2.2, 2.3, 4.2, and 4.3), to mixed systems, which are unions of several homogeneous systems (Theorems 3.3, 3.6, 4.7, and 4.10).

The next stage of our research is to study the behavior of chaoses in arbitrary symmetric spaces. The class of symmetric spaces in which the homogeneous Rademacher chaos forms an unconditional sequence is characterized in papers [27; 28]. However, even for the special case of Rademacher chaos, a similar question regarding the property of random unconditional convergence remains open.

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