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REDUCIBLE τ-CLOSED σ-LOCAL FORMATIONS OF FINITE GROUPS WITH A GIVEN STRUCTURE OF SUBFORMATIONS

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Abstract. Let $\mathfrak{F}$ and $\mathfrak{H}$ be some τ-closed σ-local formations of finite groups. By $\mathfrak{F}/^{\tau}_{\sigma}\mathfrak{H}\cap\mathfrak{F}$ we denote the lattice of all τ-closed σ-local formations $\mathfrak{X}$ such that $\mathfrak{H}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathfrak{F}$. The length of the lattice $\mathfrak{F}/^{\tau}_{\sigma}\mathfrak{H}\cap\mathfrak{F}$ is called the $\mathfrak{H}^{\tau}_{\sigma}$-*defect*, and for $\mathfrak{H}=(1)$ it is the formation of all identity groups, l^{τ}_{σ}-*length* of $\mathfrak{F}$. The general properties of the $\mathfrak{H}^{\tau}_{\sigma}$-defect of τ-closed σ-local formations are studied, and a description of the structural structure of reducible τ-closed σ-local formations with $\mathfrak{H}^{\tau}_{\sigma}$-defect $\leqslant 2$ and l^{τ}_{σ}-length $\leqslant 3$ is obtained.

ПРИВОДИМЫЕ τ-ЗАМКНУТЫЕ σ-ЛОКАЛЬНЫЕ ФОРМАЦИИ КОНЕЧНЫХ ГРУПП С ЗАДАННОЙ СТРУКТУРОЙ ПОДФОРМАЦИЙ

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Ключевые слова: конечная группа, подгрупповой функтор, τ-замкнутая σ-локальная формация, критическая τ-замкнутая σ-локальная формация, $\mathfrak{H}^{\tau}_{\sigma}$-дефект формации, l^{τ}_{σ}-длина формации.

Аннотация. Пусть $\mathfrak{F}$ и $\mathfrak{H}$ – некоторые τ-замкнутые σ-локальные формации конечных групп. Через $\mathfrak{F}/^{\tau}_{\sigma}\mathfrak{H}\cap\mathfrak{F}$ обозначают решетку всех τ-замкнутых σ-локальных формаций $\mathfrak{X}$ таких, что $\mathfrak{H}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathfrak{F}$. Длину решетки $\mathfrak{F}/^{\tau}_{\sigma}\mathfrak{H}\cap\mathfrak{F}$ называют $\mathfrak{H}^{\tau}_{\sigma}$-*дефектом*, а при $\mathfrak{H}=(1)$ – формация всех единичных групп, l^{τ}_{σ}-*длиной* формации $\mathfrak{F}$. Изучены общие свойства $\mathfrak{H}^{\tau}_{\sigma}$-дефекта τ-замкнутых σ-локальных формаций, получено описание структурного строения приводимых τ-замкнутых σ-локальных формаций, имеющих $\mathfrak{H}^{\tau}_{\sigma}$-дефект $\leqslant 2$ и l^{τ}_{σ}-длину $\leqslant 3$.

1. Introduction

All groups under consideration are finite. We adhere to the terminology and notation adopted in [1–4].

The study and classification of formations with given restrictions on the lattices of their subformations is one of the most interesting and meaningful problems in the theory of formations of finite groups.

In 1986, A. N. Skiba [5] proved that the lattice of all formations, as well as the lattice of all local formations, are modular. This result made it possible to apply the methods and constructions of general lattice theory to the study of the structural structure of formations of finite groups. The study of the structural structure of a local formation $\mathfrak{F}$ based on the properties of its well-studied subformation was first carried out by A. N. Skiba and E. A. Targonskii [6]. This approach was based on their concept of the $\mathfrak{H}$-defect of a local formation. In the paper [6], the basic properties of the $\mathfrak{H}$-defect of a local formation were studied, and a classification of local formations of nilpotent defect $\leqslant 2$ was obtained. Subsequently, this method was widely used in studying the structural structure of not only local formations, but also formations of other types, such as τ-closed multiply and totally local formations, partially saturated and partially composition formations, etc. Moreover, $\mathfrak{H}$ was considered not only as the formation of all nilpotent groups, but also other fairly well-known classes (the class of all π-decomposable, π-nilpotent, metanilpotent, soluble, supersoluble groups, etc.).

In this paper, we study the structural structure of τ-closed σ-local formations based on the ideas and results of [2; 6]. Following [2; 6], we introduce the concept of the $\mathfrak{H}^{\tau}_{\sigma}$-defect of a τ-closed σ-local formation, as well as the l^{τ}_{σ}-length of a τ-closed σ-local formation, study the basic properties of the $\mathfrak{H}^{\tau}_{\sigma}$-defect of a formation, and investigate the structural structure of τ-closed σ-local formations of finite $\mathfrak{H}^{\tau}_{\sigma}$-defect and l^{τ}_{σ}-length.

The following main results are obtained in the paper: a description of minimal τ-closed σ-local not $\mathfrak{H}$-formations for an arbitrary τ-closed σ-local σ-nilpotent formation $\mathfrak{H}$, i. e. irreducible τ-closed σ-local formations of $\mathfrak{H}^{\tau}_{\sigma}$-defect 1 is given; the existence of $\mathfrak{H}^{\tau}_{\sigma}$-critical formations for every τ-closed σ-local formation $\mathfrak{F} \not\subseteq \mathfrak{H}$ is proved; a characterization of τ-closed σ-local formations of $\mathfrak{H}^{\tau}_{\sigma}$-defect 1 is obtained; a description of the structure of τ-closed σ-local formations of $\mathfrak{H}^{\tau}_{\sigma}$-defect $\leqslant 2$ and l^{τ}_{σ}-length $\leqslant 3$ is given.

We prove the main results of the paper in Sections 3–7 and also consider some of the most interesting consequences of the obtained results.

2. Basic definitions and some auxiliary results

The basic concepts of the theory of σ-properties of groups, as well as general properties of τ-closed σ-local formations and their lattices are presented in the papers [1; 4; 7–23].

Let σ be some partition of the set of all primes $\mathbb{P}$, i. e. $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$, G be a group, and $\mathfrak{F}$ be a class of groups. Then $\sigma(G) = \{\sigma_i \mid \sigma_i \bigcap \pi(G) \neq \varnothing\}$ and $\sigma(\mathfrak{F}) = \bigcup_{G\in\mathfrak{F}} \sigma(G)$.

The group G is called [1]: *σ-primary* if G is a σ_i-group for some i; *σ-nilpotent* if every chief factor H/K from G is σ-central in G, that is, the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary; *σ-soluble* if $G = 1$ or $G \neq 1$ and each chief factor from G is σ-primary.

The symbol $\mathfrak{S}_\sigma$ denotes the class of all σ-soluble groups and $\mathfrak{N}_\sigma$ denotes the class of all σ-nilpotent groups. For any $\sigma_i \in \sigma$ the symbol $\mathfrak{G}_{\sigma_i}$ denotes the class of all σ_i-groups.

Recall that a class of groups $\mathfrak{F}$ is called *a formation* if: 1) $G/N \in \mathfrak{F}$ when $G \in \mathfrak{F}$, and 2) $G/N \cap K \in \mathfrak{F}$ when $G/N \in \mathfrak{F}$ and $G/K \in \mathfrak{F}$.

Every function f of the form $f : \sigma \to \{\text{formations of groups}\}$ is called *a formation σ-function* [4]. For any formation σ-function f the class $LF_\sigma(f)$ defined as follows:

$$LF_\sigma(f) = (G \mid G = 1 \text{ or } G \neq 1 \text{ and } G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then the class $\mathfrak{F}$ is called *σ-local*, and f called *σ-local definition* of $\mathfrak{F}$.

Let $\tau(G)$ be a set of subgroups of G such that $G \in \tau(G)$. Then τ is called a *subgroup functor* [2] if for every epimorphism $\varphi : A \to B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

The subgroup functor τ is called [2]: *trivial*, if for any group G we have $\tau(G) = \{G\}$; *identity*, if for any group G we have $\tau(G) = s(G)$ is the collection of all subgroups of G.

A formation $\mathfrak{F}$ is called *τ-closed*, if $\tau(G) \subseteq \mathfrak{F}$ for any group $G \in \mathfrak{F}$. In particular, a formation is called: *hereditary*, if it is τ-closed, where $\tau = s$ is a identity subgroup functor; *normally hereditary*, if its is τ-closed, where $\tau(G) = s_n(G)$ is the collection of all normal subgroups of G for any group G.

The collection of all τ-closed σ-local formations denote by l^{τ}_{σ}. Formations from l^{τ}_{σ} we call *l^{τ}_{σ}-formations.* In particular, if τ is a trivial subgroup functor [2], that is $\tau(G) = \{G\}$ for all G, the symbol τ we omits and denotes by l_σ the collection of all σ-local formations.

If f is a formation σ-function, then the symbol $\mathrm{Supp}(f)$ denotes the support of f, that is, the set of all σ_i such that $f(\sigma_i) \neq \varnothing$. A formation σ-function f is called: *τ-valued*, if $f(\sigma_i)$ is τ-closed formation for each $\sigma_i \in \mathrm{Supp}(f)$; *integrated* if $f(\sigma_i) \subseteq LF_\sigma(f)$ for all i; *full* if $f(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all i. If F is a full integrated formation σ-function and $\mathfrak{F} = LF_\sigma(F)$, then F is called the *canonical σ-local definition* of $\mathfrak{F}$.

We also use $\cap_{j\in J} f_j$ to denote a formation σ-function h such that $h(\sigma_i) = \cap_{j\in J} f_j(\sigma_i)$, in particular, $h(\sigma_i) = (f_1 \cap f_2)(\sigma_i) = f_1(\sigma_i) \cap f_2(\sigma_i)$, for all i.

Let $\{f_j \mid j \in J\}$ be a set of all τ-valued σ-local definitions of $\mathfrak{F}$. Then we say that $f = \cap_{j\in J} f_j$ is *the smallest τ-valued σ-local definition* of $\mathfrak{F}$.

For any set of groups $\mathfrak{X}$ the symbol $l^{\tau}_{\sigma}\mathrm{form}\,\mathfrak{X}$ denotes *a τ-closed σ-local formation generated by* $\mathfrak{X}$, that is, $l^{\tau}_{\sigma}\mathrm{form}\,\mathfrak{X}$ is the intersection of all τ-closed σ-local formations containing $\mathfrak{X}$. If $\mathfrak{F} = l^{\tau}_{\sigma}\mathrm{form}\,G$ for some group G, then $\mathfrak{F}$ is called a *one-generated τ-closed σ-local formation*.

Let $\{\mathfrak{F}_j \mid j \in J\}$ be some collection of τ-closed σ-local formations. Then we put $\vee^{\tau}_{\sigma}(\mathfrak{F}_j \mid j \in J) = l^{\tau}_{\sigma}\mathrm{form}\,(\cup_{j\in J}\mathfrak{F}_j)$. In particular, for any two l^{τ}_{σ}-formations $\mathfrak{M}$ and $\mathfrak{H}$ we set $\mathfrak{M} \vee^{\tau}_{\sigma} \mathfrak{H} = l^{\tau}_{\sigma}\mathrm{form}\,(\mathfrak{M} \cup \mathfrak{H})$.

For an arbitrary set of groups $\mathfrak{X}$ and any $\sigma_i \in \sigma$, the symbol $\mathfrak{X}(\sigma_i)$ [9, p. 962] denotes the class of groups defined as follows: $\mathfrak{X}(\sigma_i) = (G/O_{\sigma_i',\sigma_i}(G) \mid G \in \mathfrak{X})$, if $\sigma_i \in \sigma(\mathfrak{X})$, $\mathfrak{X}(\sigma_i) = \varnothing$, if $\sigma_i \notin \sigma(\mathfrak{X})$.

Following [24; 25], by a *minimal* τ*-closed* σ*-local not* $\mathfrak{H}$*-formation* or an $\mathfrak{H}_\sigma^\tau$*-critical formation* we mean a τ-closed σ-local formation $\mathfrak{F} \not\subseteq \mathfrak{H}$, all of whose proper τ-closed σ-local subformations are contained in the class of groups $\mathfrak{H}$.

Recall [2, p. 12] that a non-empty set of formations θ is called a *complete lattice of formations* if the intersection of any set of formations from θ again belongs to θ, and the set θ contains a formation $\mathfrak{M}$ such that $\mathfrak{H} \subseteq \mathfrak{M}$ for all $\mathfrak{H} \in \theta$. Any formation from θ is called a θ*-formation.*

For any two θ-formations $\mathfrak{M}$ and $\mathfrak{H}$, where $\mathfrak{M} \subseteq \mathfrak{H}$, we denote by $\mathfrak{H}/_\theta\mathfrak{M}$ [2, p. 168] the lattice of θ-formations $\mathfrak{X}$ such that $\mathfrak{M} \subseteq \mathfrak{X} \subseteq \mathfrak{H}$. In particular, $\mathfrak{H}/_\sigma^\tau\mathfrak{M}$ denotes the lattice of τ-closed σ-local formations $\mathfrak{X}$ such that $\mathfrak{M} \subseteq \mathfrak{X} \subseteq \mathfrak{H}$.

Let θ be some complete modular lattice of formations. For any two θ-formations $\mathfrak{F}$ and $\mathfrak{M}$ [2, p. 192], where $\mathfrak{M} \subseteq \mathfrak{F}$, $|\mathfrak{F} : \mathfrak{M}|_\theta$ denote the length of the lattice $\mathfrak{F}/_\theta\mathfrak{M}$ of θ-formations contained between $\mathfrak{M}$ and $\mathfrak{F}$. Let $\mathfrak{F}$ and $\mathfrak{H}$ be arbitrary θ-formations. Then the $\mathfrak{H}_\theta$*-defect* of the formation $\mathfrak{F}$ is the lattice length $\mathfrak{F}/_\theta\mathfrak{H} \cap \mathfrak{F}$ (finite or infinite) and is denoted by $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|_\theta$.

Let 0_θ be the zero of the lattice θ, $\mathfrak{F} \in \theta$. Then the θ*-length* [2, p. 212] of the formation $\mathfrak{F}$ is the cardinal number $|\mathfrak{F} : 0_\theta|_\theta$. In particular, the *length* of the formation $\mathfrak{F}$ is the number $l(\mathfrak{F}) = |\mathfrak{F} : \varnothing|$; the *length* of the local formation $\mathfrak{F}$ is the number $l_1(\mathfrak{F}) = |\mathfrak{F} : (1)|_l$.

Following [2, p. 192] an $\mathfrak{H}_\sigma^\tau$*-defect* of a τ-closed σ-local formation $\mathfrak{F}$, we will call the lattice length $\mathfrak{F}/_\sigma^\tau\mathfrak{H} \cap \mathfrak{F}$ and denote it by $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|_\sigma^\tau$. Similarly, following [2, p. 212], an l_σ^τ*-length* of a τ-closed σ-local formation $\mathfrak{F}$ is the number $l_\sigma^\tau(\mathfrak{F}) = |\mathfrak{F} : (1)|_\sigma^\tau$.

Let us also recall the concept of direct decomposition of a formation (see [2, p. 171]). Let $\{\mathfrak{F}_j \mid j \in J\}$ be some nonempty set of subclasses of $\mathfrak{F}_j \subseteq \mathfrak{F}$ such that $\mathfrak{F}_{j_1} \cap \mathfrak{F}_{j_2} = (1)$ for any $j_1 \neq j_2$ in J. If, in addition, every group $G \in \mathfrak{F}$ has the form $G = A_{j_1} \times \ldots \times A_{j_t}$, where $A_{j_1} \in \mathfrak{F}_{j_1}, \ldots, A_{j_t} \in \mathfrak{F}_{j_t}$ for some $j_1, \ldots, j_t \in J$, then we write that $\mathfrak{F} = \oplus_{j\in J}\mathfrak{F}_j$ (in particular, $\mathfrak{F} = \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_t$, if $J = \{1, \ldots, t\}$).

A subformation $\mathfrak{M}$ of a formation $\mathfrak{F}$ is called *complemented* [2, p. 170] in $\mathfrak{F}$ if $\mathfrak{F} = \mathrm{form}(\mathfrak{M} \cup \mathfrak{H})$ and $\mathfrak{M} \cap \mathfrak{H} = (1)$ for some subformation $\mathfrak{H}$ of $\mathfrak{F}$. In this case, the subformation $\mathfrak{H}$ is called *complement* of $\mathfrak{M}$ in $\mathfrak{F}$.

To prove the main result of the paper, we need the following known facts from formation theory.

A special case of Theorem 1.15 [16] is the following lemma.

Lemma 2.1 [16, Theorem 1.15]. *The set* l_σ^τ *of all* τ*-closed* σ*-local formations forms an algebraic modular lattice of formations.*

Lemma 2.2 [26, Chapter II, §8, Theorem 16]. *Let* $\mathcal{L}$ *be a lattice of finite length. Then the following conditions are equivalent:*

(i) *the modular law holds in* $\mathcal{L}$*;*

(ii) $\mathcal{L}$ *is upper and lower semimodular;*

(iii) $\mathcal{L}$ *satisfies the Jordan-Dedekind chain condition and* $h[x] + h[y] = h[x \vee y] + h[x \wedge y]$.

Lemma 2.3 [17, Lemma 2.1]. *Let* Π *be a nonempty subset of* σ. *Then* $\mathfrak{G}_\Pi$ *of all* Π*-groups and the class* $\mathfrak{N}_\Pi$ *of all* σ*-nilpotent* Π*-groups are* σ*-local formations and the following statements hold.*

(1) $\mathfrak{G}_\Pi = LF_\sigma(g)$, *where* g *is the canonical* σ*-local definition of the formation* $\mathfrak{G}_\Pi$. *Moreover,* $g(\sigma_i) = \mathfrak{G}_\Pi$ *for all* $\sigma_i \in \Pi$ *and* $g(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$;

(2) $\mathfrak{N}_\Pi = LF_\sigma(n) = LF_\sigma(N)$, *where* n *and* N *are, respectively, the smallest and canonical* σ*-local definitions of the formation* $\mathfrak{N}_\Pi$. *Moreover,* $n(\sigma_i) = (1)$ *for all* $\sigma_i \in \Pi$ *and* $n(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$, $N(\sigma_i) = \mathfrak{G}_{\sigma_i}$ *for all* $\sigma_i \in \Pi$ *and* $N(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$.

Lemma 2.4 [21, Theorem]. *Let* $\mathfrak{F}$ *be a nonempty formation. Then the following statements are equivalent:*

(i) $\mathfrak{F}$ *is* τ*-closed n-multiply* σ*-local* $(n \geqslant 1)$;

(ii) $\mathfrak{G}_{\sigma_i}\mathfrak{F}^\tau_{\sigma_{n-1}}(\sigma_i) \subseteq \mathfrak{F}$ *for all* $\sigma_i \in \sigma(\mathfrak{F})$;

(iii) $\mathfrak{F} = \mathrm{form}(\cup_{\sigma_i\in\sigma(\mathfrak{F})}\mathfrak{G}_{\sigma_i}\mathfrak{F}^\tau_{\sigma_{n-1}}(\sigma_i))$.

Lemma 2.5 [15, p. 2372]. *Let* $\mathfrak{F} = \oplus_{j\in J}\mathfrak{F}_j$, *where* $\{\mathfrak{F}_j \mid j \in J\}$ *is the set of formations such that* $\sigma(\mathfrak{F}_a) \cap \sigma(\mathfrak{F}_b) = \varnothing$ *for any* $a, b \in J$, $a \neq b$. *If and only if the formation* $\mathfrak{F}$ *is n-multiply* σ*-local* $(n \geqslant 1)$, $\mathfrak{F}_j$ *is n-multiply* σ*-local formation for all* j.

Lemma 2.6 [26, Ch. II, §7, Theorem 12]. *If* a, b, c *are elements of the modular lattice* $\mathcal{M}$, *then if either of the two equalities* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *or* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ *holds, the triple* $\{a, b, c\}$ *is distributive.*

Lemma 2.7 is a special case of Lemma 2.6 [9].

Lemma 2.7 [9, Lemma 2.6]. *Let $\mathfrak{F} = l_\sigma^\tau \mathrm{form}(\mathfrak{X}) = LF_\sigma(f)$ – τ-closed σ-local formation generated by $\mathfrak{X}$ and $\Pi = \sigma(\mathfrak{X})$. Let m be a formation σ-function such that $m(\sigma_i) = \tau\mathrm{form}(\mathfrak{X}(\sigma_i))$ for all $\sigma_i \in \Pi$ and $m(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Then:*

(1) $\Pi = \sigma(\mathfrak{F})$;

(2) *m is a τ-valued σ-local definition of $\mathfrak{F}$; and*

(3) *$m(\sigma_i) \subseteq f(\sigma_i) \cap \mathfrak{F}$ for all i.*

The following lemma is a special case of Lemma 3.1 [16].

Lemma 2.8 [16, Lemma 3.1]. *Let $\mathfrak{F}_j = LF_\sigma(f_j)$ for all $j \in J$, where f_j is the τ-valued σ-local definition of $\mathfrak{F}_j$, $\mathfrak{F} = \cap_{j\in J}\mathfrak{F}_j$, and $f = \cap_{j\in J} f_j$. Then:*

(1) $\sigma(\mathfrak{F}) = \cap_{j\in J}\sigma(\mathfrak{F}_j) = \mathrm{Supp}(f)$*;*

(2) *$\mathfrak{F} = LF_\sigma(f)$ is a τ-closed σ-local formation, where f is a τ-valued formation σ-function.*

Furthermore, if f_j is an integrated τ-valued formation σ-function for all $j \in J$, then f is also an integrated τ-valued formation σ-function.

Lemma 2.9 [23, Theorem]. *Let $\mathfrak{H}$ be a σ-local formation of classical type and h be its canonical σ-local definition. Then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) *$G = P$ is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) = \{1, G\}$;*

2) *P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$;*

3) *$G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is either a monolithic $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with monolith $Q = K^{h(\sigma_i)} \nsubseteq \Phi(K)$, where $\sigma_i \notin \sigma(Q)$, or a minimal non-$h(\sigma_i)$-group of one of the following types:* a) *the quaternion group of order* 8, *if* $2 \notin \sigma_i$; b) *an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$*; c) *a cyclic q-group, $q \notin \sigma_i$.*

Lemma 2.10 [9, Lemma 2.1]. *Let f and h be formation σ-functions and let $\Pi = \mathrm{Supp}(f)$. Let us assume that $\mathfrak{F} = LF_\sigma(f) = LF_\sigma(h)$. Then:*

(1) $\Pi = \sigma(\mathfrak{F})$;

(2) *$\mathfrak{F} = (\cap_{\sigma_i\in\Pi}\mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{G}_\Pi$. Therefore, $\mathfrak{F}$ is a saturated formation;*

(3) *If every group in $\mathfrak{F}$ is σ-soluble, then $\mathfrak{F} = (\cap_{\sigma_i\in\Pi}\mathfrak{S}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{S}_\Pi$;*

(4) *If $\sigma_i \in \Pi$, then $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{F}$;*

(5) *$\mathfrak{F} = LF_\sigma(F)$, where F is the unique formation σ-function such that $F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}$ for all $\sigma_i \in \Pi$ and $F(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Furthermore, $F(\sigma_i) = \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F})$ for all i.*

Lemma 2.11 is a special case of Corollary 3.1 [14].

Lemma 2.11 [14, Corollary 3.1]. *Let f_j be the smallest τ-valued σ-local definition of $\mathfrak{F}_j$, $j = 1, 2$. Then $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ if and only if $f_1 \leqslant f_2$.*

Lemma 2.12 [1, Lemma 18.8]. *If a group G has only one minimal normal subgroup and $O_p(G) = 1$ (p is some prime number), then there exists a faithful irreducible F_pG-module, where F_p is a field of p elements.*

Lemma 2.13 [16, Corollary 3.7]. *For any σ-localformations $\mathfrak{M}$ and $\mathfrak{H}$, there is a lattice isomorphism $\mathfrak{M} \vee_\sigma^\tau \mathfrak{H} /_\sigma^\tau \mathfrak{M} \simeq \mathfrak{H} /_\sigma^\tau \mathfrak{H} \cap \mathfrak{M}$.*

Lemma 2.14 [2, Theorem 4.3.2]. *Let $\mathfrak{M}$ be a nonempty subformation of $\mathfrak{F}$. Then if $\mathfrak{H}$ is the complement of $\mathfrak{M}$ in $\mathfrak{F}$, then $\mathfrak{F} = \{A \times B \mid A \in \mathfrak{M}, B \in \mathfrak{H}\}$.*

3. $\mathfrak{H}_\sigma^\tau$-defect formation

Let $\mathfrak{H}$ and $\mathfrak{F}$ be τ-closed σ-local formations. Following [2, p. 192] an $\mathfrak{H}_\sigma^\tau$-*defect* of $\mathfrak{F}$ we will call the lattice length $\mathfrak{F} /_\sigma^\tau \mathfrak{H} \cap \mathfrak{F}$ and denote it by $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|_\sigma^\tau$.

By Lemma 2.1 the following two statements are special cases (for $\theta = l_\sigma^\tau$) of Lemmas 5.2.8 and 5.2.7 [2], respectively.

Lemma 3.1. *Let $\mathfrak{M}$, $\mathfrak{F}$, $\mathfrak{X}$, and $\mathfrak{H}$ be τ-closed σ-local formations, and $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{X}$. Then if m, r, and t are, respectively, $\mathfrak{H}_\sigma^\tau$-defects of the formations $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$, and m, $r < \infty$, then $t \leqslant m + r$.*

Lemma 3.2. *Let $\mathfrak{M}$, $\mathfrak{F}$, and $\mathfrak{H}$ be τ-closed σ-local formations, and $\mathfrak{M}\subseteq\mathfrak{F}$. Then* $|\mathfrak{M}:\mathfrak{H}\cap\mathfrak{M}|_\sigma^\tau \leqslant |\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau$.

An element a of the lattice $\mathfrak{L}$ is called *neutral* (otherwise *distributive*) [26, p. 96], if for any $b,\ c\in\mathfrak{L}$ the triple a,b,c generates a distributive sublattice in the lattice $\mathfrak{L}$.

Lemma 3.3. *Let $\mathfrak{M}$ and $\mathfrak{F}$ be τ-closed σ-local formations of finite $\mathfrak{H}_\sigma^\tau$-defect, where $\mathfrak{H}$ is the neutral element of the lattice of τ-closed σ-local formations. Then for the $\mathfrak{H}_\sigma^\tau$-defect of the formation $\mathfrak{M}\vee_\sigma^\tau\mathfrak{F}$ the following equality holds:*

$$|\mathfrak{M}\vee_\sigma^\tau\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\vee_\sigma^\tau\mathfrak{F})|_\sigma^\tau = |\mathfrak{M}:\mathfrak{H}\cap\mathfrak{M}|_\sigma^\tau + |\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau - |\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau.$$

Proof. Let $\mathfrak{M}$, $\mathfrak{F}$, and $\mathfrak{H}$ be σ-local formations satisfying the condition of the lemma. Let $\mathfrak{X} = \mathfrak{M}\vee_\sigma^\tau\mathfrak{F}$, $\mathfrak{L}=\mathfrak{M}\cap\mathfrak{F}$, $t=|\mathfrak{X}:\mathfrak{H}\cap\mathfrak{X}|_\sigma^\tau$, $m=|\mathfrak{M}:\mathfrak{H}\cap\mathfrak{M}|_\sigma^\tau$, $k=|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau$ and $l=|\mathfrak{L}:\mathfrak{H}\cap\mathfrak{L}|_\sigma^\tau$. By Lemma 3.1, we have $t\leqslant m+k$.

Let now $\mathfrak{X}_1 := \mathfrak{X}\vee_\sigma^\tau\mathfrak{H}$, $\mathfrak{M}_1 := \mathfrak{M}\vee_\sigma^\tau\mathfrak{H}$, $\mathfrak{F}_1 := \mathfrak{F}\vee_\sigma^\tau\mathfrak{H}$ and $\mathfrak{L}_1 := \mathfrak{L}\vee_\sigma^\tau\mathfrak{H}$. By virtue of Lemmas 3.1 and 3.2, equality $|\mathfrak{X}_1:\mathfrak{H}\cap\mathfrak{X}_1|_\sigma^\tau = t$, $|\mathfrak{M}_1:\mathfrak{H}\cap\mathfrak{M}_1|_\sigma^\tau=m$, $|\mathfrak{F}_1:\mathfrak{H}\cap\mathfrak{F}_1|_\sigma^\tau=k$ and $|\mathfrak{L}_1:\mathfrak{H}\cap\mathfrak{L}_1|_\sigma^\tau=l$. Therefore, the lattice length $\mathfrak{X}_1/_\sigma^\tau(\mathfrak{X}_1\cap\mathfrak{H}) = \mathfrak{X}\vee_\sigma^\tau\mathfrak{H}/_\sigma^\tau\mathfrak{H}$ is equal to t. Note also that the formations $\mathfrak{M}_1$ and $\mathfrak{F}_1$ are elements of the lattice $\mathfrak{X}\vee_\sigma^\tau\mathfrak{H}/_\sigma^\tau\mathfrak{H}\simeq\mathfrak{X}/_\sigma^\tau\mathfrak{X}\cap\mathfrak{H}$ and an $\mathfrak{H}_\sigma^\tau$-defect is a function of the lattice height $\mathfrak{X}/_\sigma^\tau\mathfrak{X}\cap\mathfrak{H}$. Therefore, by Lemma 2.2 the following holds equality

$$|\mathfrak{M}_1\vee_\sigma^\tau\mathfrak{F}_1:\mathfrak{H}\cap(\mathfrak{M}_1\vee_\sigma^\tau\mathfrak{F}_1)|_\sigma^\tau = |\mathfrak{M}_1:\mathfrak{H}\cap\mathfrak{M}_1|_\sigma^\tau + |\mathfrak{F}_1:\mathfrak{H}\cap\mathfrak{F}_1|_\sigma^\tau - |\mathfrak{M}_1\cap\mathfrak{F}_1:\mathfrak{H}\cap(\mathfrak{M}_1\cap\mathfrak{F}_1)|_\sigma^\tau. \quad (*)$$

Because $\mathfrak{M}_1\vee_\sigma^\tau\mathfrak{F}_1 = (\mathfrak{M}\vee_\sigma^\tau\mathfrak{H})\vee_\sigma^\tau(\mathfrak{F}\vee_\sigma^\tau\mathfrak{H}) = \mathfrak{X}\vee_\sigma^\tau\mathfrak{H}$, that $|\mathfrak{M}_1\vee_\sigma^\tau\mathfrak{F}_1:\mathfrak{H}\cap(\mathfrak{M}_1\vee_\sigma^\tau\mathfrak{F}_1)|_\sigma^\tau = t$. Furthermore, by hypothesis, $\mathfrak{H}$ is a neutral element of the lattice of τ-closed σ-local formations, therefore $\mathfrak{M}_1\cap\mathfrak{F}_1 = (\mathfrak{M}\vee_\sigma^\tau\mathfrak{H})\cap(\mathfrak{F}\vee_\sigma^\tau\mathfrak{H}) = (\mathfrak{M}\cap\mathfrak{F})\vee_\sigma^\tau\mathfrak{H} = \mathfrak{L}\vee_\sigma^\tau\mathfrak{H} = \mathfrak{L}_1$.

Finally, since $|\mathfrak{L}_1:\mathfrak{H}\cap\mathfrak{L}_1|_\sigma^\tau = l = |\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau$, then from $(*)$ we get

$$|\mathfrak{M}\vee_\sigma^\tau\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\vee_\sigma^\tau\mathfrak{F})|_\sigma^\tau = |\mathfrak{M}:\mathfrak{H}\cap\mathfrak{M}|_\sigma^\tau + |\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau - |\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau. \qquad \square$$

Lemma 3.4. *Let $\mathfrak{F}$, $\mathfrak{M}$, and $\mathfrak{H}$ be τ-closed σ-local formations such that $\mathfrak{H}\subseteq\mathfrak{M}$. Then the $\mathfrak{H}_\sigma^\tau$-defect of $\mathfrak{F}$ is finite if and only if the $\mathfrak{H}_\sigma^\tau$-defect of the formation $\mathfrak{M}\cap\mathfrak{F}$ and $\mathfrak{M}_\sigma^\tau$-defect formations $\mathfrak{F}$, in this case*

$$|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau = |\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau + |\mathfrak{F}:\mathfrak{M}\cap\mathfrak{F}|_\sigma^\tau.$$

Proof. Necessity. Assume that the $\mathfrak{H}_\sigma^\tau$-defect of $\mathfrak{F}$ is finite and let $|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau = n$. Then, by Lemma 3.2, the inequality $|\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau \leqslant |\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau$. Therefore, the $\mathfrak{H}_\sigma^\tau$-defect $\mathfrak{M}\cap\mathfrak{F}$ is also finite. Let $k=|\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau$. By the definition of $\mathfrak{H}_\sigma^\tau$-defect and by Lemma 2.1, the modularity of the lattice l_σ^τ, implies that there exist chains

$$\mathfrak{H}\cap\mathfrak{F} = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \ldots \subset \mathfrak{F}_{n-1} \subset \mathfrak{F}_n = \mathfrak{F},$$

$$\mathfrak{H}\cap\mathfrak{F} = \mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F}) = \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \ldots \subset \mathfrak{L}_{k-1} \subset \mathfrak{L}_k = \mathfrak{M}\cap\mathfrak{F}$$

from $\mathfrak{H}\cap\mathfrak{F}$ to $\mathfrak{F}$ and $\mathfrak{M}\cap\mathfrak{F}$ respectively, which $\mathfrak{F}_i$ is the maximal τ-closed σ-local subformation in $\mathfrak{F}_{i+1}$ and $\mathfrak{L}_j$ is the maximal τ-closed σ-local subformation in $\mathfrak{L}_{j+1}$ for all $i=0,1,\ldots,n-1$ and $j=0,1,\ldots,k-1$. Since $\mathfrak{H}\cap\mathfrak{F}\subseteq\mathfrak{M}\cap\mathfrak{F}\subseteq\mathfrak{F}$, then, by Lemma 2.1 from the modularity of the lattice l_σ^τ It follows that there exists a chain $\mathfrak{M}\cap\mathfrak{F} = \mathfrak{X}_0 \subset \mathfrak{X}_1 \subset \ldots \subset \mathfrak{X}_{t-1} \subset \mathfrak{X}_t = \mathfrak{F}$ of length $t=n-k$ such that $\mathfrak{X}_i$ is the maximal τ-closed σ-local subformation in $\mathfrak{X}_{i+1}$, $i=0,1,\ldots,t-1$. Therefore, the lattice $\mathfrak{F}/_\sigma^\tau\mathfrak{M}\cap\mathfrak{F}$ has finite length equal to t. Then $t=|\mathfrak{F}:\mathfrak{M}\cap\mathfrak{F}|_\sigma^\tau$ by the definition of the $\mathfrak{M}_\sigma^\tau$-defect.

Sufficiency. Let $k=|\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau$ and $t=|\mathfrak{F}:\mathfrak{M}\cap\mathfrak{F}|_\sigma^\tau$. Then we have

$$\mathfrak{M}\cap\mathfrak{F} = \mathfrak{X}_0 \subset \mathfrak{X}_1 \subset \ldots \subset \mathfrak{X}_{t-1} \subset \mathfrak{X}_t = \mathfrak{F},$$

$$\mathfrak{H}\cap\mathfrak{F} = \mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F}) = \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \ldots \subset \mathfrak{L}_{k-1} \subset \mathfrak{L}_k = \mathfrak{M}\cap\mathfrak{F},$$

where $\mathfrak{X}_i$ and $\mathfrak{L}_j$ are the maximal τ-closed σ-local subformation in $\mathfrak{X}_{i+1}$ and $\mathfrak{L}_{j+1}$, respectively, $i = 0,1,\ldots,t-1$ and $j=0,1,\ldots,k-1$. Therefore, there exists a maximal chain of τ-closed σ-local formations of length $k+t$ from $\mathfrak{H}\cap\mathfrak{F}$ to $\mathfrak{F}$. By Lemma 2.1, the latter implies that $|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau = k+t$, i. e.

$$|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_\sigma^\tau = |\mathfrak{M}\cap\mathfrak{F}:\mathfrak{H}\cap(\mathfrak{M}\cap\mathfrak{F})|_\sigma^\tau + |\mathfrak{F}:\mathfrak{M}\cap\mathfrak{F}|_\sigma^\tau. \qquad \square$$

Lemma 3.5. *Let $\mathfrak{H}$ be a τ-closed σ-local formation such that $(1)\neq\mathfrak{H}\subset\mathfrak{N}_\sigma$. Then $\mathfrak{H}=\mathfrak{N}_{\sigma(\mathfrak{H})}$.*

Proof. Let $\Pi = \sigma(\mathfrak{H})$ and $\mathfrak{G}_\Pi$ be the class of all Π-groups. By Lemma 2.3(1), the formation $\mathfrak{G}_\Pi$ is σ-local. Moreover, since the formation $\mathfrak{G}_\Pi$ is hereditary, it is τ-closed for any subgroup functor τ. Therefore, the inclusion $\mathfrak{H} \subseteq \mathfrak{G}_\Pi \cap \mathfrak{N}_\sigma = \mathfrak{N}_\Pi$.

On the other hand, in view of Lemma 2.4(ii), we have $\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{H}^\tau_{\sigma_0}(\sigma_i) \subseteq \mathfrak{H}$ for all $\sigma_i \in \Pi$. Thus, taking into account Lemma 2.5 we have $\mathfrak{N}_\Pi = \oplus_{\sigma_i \in \Pi}\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{H}$. Thus, $\mathfrak{H} = \mathfrak{N}_\Pi$, where $\Pi = \sigma(\mathfrak{H})$. □

Lemma 3.6. *Every σ-nilpotent τ-closed σ-local formation is a neutral element of the lattice l^τ_σ. In particular, the formation $\mathfrak{N}_\sigma$ of all σ-nilpotent groups is a neutral element of the lattice l^τ_σ.*

Proof. Let $\mathfrak{H}$, $\mathfrak{F}$, and $\mathfrak{M}$ be some τ-closed σ-local formations, where $\mathfrak{H}$ is σ-nilpotent. By Lemmas 2.1 and 2.6, to prove the assertion of the lemma, it suffices to show that $\mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}) = (\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M})$.

If $\mathfrak{H} = (1)$, then the statement is obvious. Let $\mathfrak{H} \neq (1)$ and $\Pi_1 = \sigma(\mathfrak{H} \cap \mathfrak{F})$ and $\Pi_2 = \sigma(\mathfrak{H} \cap \mathfrak{M})$. Since $(\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M}) = l^\tau_\sigma\mathrm{form}((\mathfrak{H} \cap \mathfrak{F}) \cup (\mathfrak{H} \cap \mathfrak{M}))$, then by Lemma 2.7(1) we have

$$\sigma((\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M})) = \sigma((\mathfrak{H} \cap \mathfrak{F}) \cup (\mathfrak{H} \cap \mathfrak{M})) = \sigma(\mathfrak{H} \cap \mathfrak{F}) \cup \sigma(\mathfrak{H} \cap \mathfrak{M}) = \Pi_1 \cup \Pi_2.$$

Since $(\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M}) \subseteq \mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M})$, then $\sigma((\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M})) \subseteq \sigma(\mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}))$, i. e. $\Pi_1 \cup \cup \Pi_2 \subseteq \sigma(\mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}))$.

On the other hand, $\mathfrak{F} \vee^\tau_\sigma \mathfrak{M} = l^\tau_\sigma\mathrm{form}(\mathfrak{F} \cup \mathfrak{M})$ and again by Lemma 2.7(1) we have

$$\sigma(\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}) = \sigma(\mathfrak{F} \cup \mathfrak{M}) = \sigma(\mathfrak{F}) \cup \sigma(\mathfrak{M}).$$

By Lemma 2.8(2), the formation $\mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M})$ is a τ-closed σ-local formation. Now, if $\sigma_i \in \sigma(\mathfrak{H} \cap \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}))$, then $\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M})$ by Lemma 2.4(ii). Therefore,

$$\sigma_i \in \sigma(\mathfrak{H}) \cap \sigma(\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}) = \sigma(\mathfrak{H}) \cap (\sigma(\mathfrak{F}) \cup \sigma(\mathfrak{M})).$$

Hence, $\mathfrak{G}_{\sigma_i} \subseteq (\mathfrak{H} \cap \mathfrak{F}) \cup (\mathfrak{H} \cap \mathfrak{M})$. Therefore,

$$\sigma_i \in \sigma(\mathfrak{H} \cap \mathfrak{F}) \cup \sigma(\mathfrak{H} \cap \mathfrak{M}) = \Pi_1 \cup \Pi_2.$$

Thus, $\sigma((\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M})) = \sigma(\mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M}))$. Since in this case both formations $(\mathfrak{H} \cap \mathfrak{F}) \vee \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M})$ and $\mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M})$ are σ-nilpotent τ-closed σ-local formations, then by Lemma 3.5 we have $(\mathfrak{H} \cap \mathfrak{F}) \vee^\tau_\sigma (\mathfrak{H} \cap \mathfrak{M}) = \mathfrak{N}_\Pi = \mathfrak{H} \cap (\mathfrak{F} \vee^\tau_\sigma \mathfrak{M})$, where $\Pi = \Pi_1 \cup \Pi_2$. Therefore, τ-closed σ-local formations $\mathfrak{H}$, $\mathfrak{F}$, and $\mathfrak{M}$ form a distributive triple in the lattice l^τ_σ, and therefore $\mathfrak{H}$ is the identity element of l^τ_σ. In particular, if $\mathfrak{H} = \mathfrak{N}_\sigma$, we obtain the second part of the lemma. □

The next lemma is a direct consequence of Lemmas 3.3 and 3.6.

Lemma 3.7. *Let $\mathfrak{M}$ and $\mathfrak{F}$ be τ-closed σ-local formations of finite $\mathfrak{H}^\tau_\sigma$-defect, where $\mathfrak{H}$ is a σ-nilpotent τ-closed σ-local formation. Then, for $\mathfrak{H}^\tau_\sigma$-defect of the formation $\mathfrak{M} \vee^\tau_\sigma \mathfrak{F}$, we have*

$$|\mathfrak{M} \vee^\tau_\sigma \mathfrak{F} : \mathfrak{H} \cap (\mathfrak{M} \vee^\tau_\sigma \mathfrak{F})|^\tau_\sigma = |\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|^\tau_\sigma + |\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|^\tau_\sigma - |\mathfrak{M} \cap \mathfrak{F} : \mathfrak{H} \cap (\mathfrak{M} \cap \mathfrak{F})|^\tau_\sigma.$$

In particular, if $\mathfrak{H} = \mathfrak{N}_\sigma$, then for the σ-nilpotent l^τ_σ-defect of the formation $\mathfrak{M} \vee^\tau_\sigma \mathfrak{F}$ we have

$$|\mathfrak{M} \vee^\tau_\sigma \mathfrak{F} : \mathfrak{N}_\sigma \cap (\mathfrak{M} \vee^\tau_\sigma \mathfrak{F})|^\tau_\sigma = |\mathfrak{M} : \mathfrak{N}_\sigma \cap \mathfrak{M}|^\tau_\sigma + |\mathfrak{F} : \mathfrak{N}_\sigma \cap \mathfrak{F}|^\tau_\sigma - |\mathfrak{M} \cap \mathfrak{F} : \mathfrak{N}_\sigma \cap (\mathfrak{M} \cap \mathfrak{F})|^\tau_\sigma.$$

4. l^τ_σ-Formations of $\mathfrak{H}^\tau_\sigma$-defect 1

Let $\mathfrak{F}$ be a τ-closed σ-local formation. Following [2, p. 200], a formation $\mathfrak{F}$ will be called an *irreducible τ-closed σ-local formation* (or an *l^τ_σ-irreducible formation*) if $\mathfrak{F} \neq l^\tau_\sigma\mathrm{form}(\cup_{i \in I}\mathfrak{X}_i) = \vee^\tau_\sigma(\mathfrak{X}_i \mid i \in \in I)$, where $\{\mathfrak{X}_i \mid i \in I\}$ is the set of all proper τ-closed σ-local subformations of $\mathfrak{F}$. If there exist such proper τ-closed σ-local subformations $\mathfrak{X}$ and $\mathfrak{H}$ of $\mathfrak{F}$, such that $\mathfrak{F} = \mathfrak{X} \vee^\tau_\sigma \mathfrak{H}$, then the formation $\mathfrak{F}$ will be called a *reducible τ-closed σ-local* (or an *l^τ_σ-reducible*) formation.

Since every minimal τ-closed σ-local non-$\mathfrak{H}$-formation $\mathfrak{F}$ is obviously an l^τ_σ-irreducible formation and its unique maximal τ-closed σ-local subformation is contained in $\mathfrak{H}$, the $\mathfrak{H}^\tau_\sigma$-defect of the formation $\mathfrak{F}$ is equal to 1. Thus, every $\mathfrak{H}^\tau_\sigma$-critical formation is an l^τ_σ-irreducible formation of $\mathfrak{H}^\tau_\sigma$-defect 1.

Theorem 4.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed σ-local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$. If and only if $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation, $\mathfrak{F} = l^\tau_\sigma\mathrm{form}\, G$ and one of the following conditions holds:*

(1) *G is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) = \{1, G\}$;*

(2) *G is a simple non-σ-primary τ-minimal non-$\mathfrak{G}_{\sigma_i}$-group for any $\sigma_i \in \sigma(G)$, $\sigma(G) \subseteq \sigma(\mathfrak{H})$;*

(3) $G = P \rtimes K$, *where* $P = C_G(P)$ *is an abelian* p*-group for some* $p \in \sigma_i \in \sigma(\mathfrak{H})$, *and* K *is a simple* σ_j*-group* ($j \neq i$) *such that* $\tau(K) = \{1, K\}$.

Proof. Necessity. Let $\mathfrak{F}$ be a minimal τ-closed σ-local non- $\mathfrak{H}$-formation. By Lemma 3.5, we have $\mathfrak{H} = \mathfrak{N}_\Pi$, where $\Pi = \sigma(\mathfrak{H})$. By Lemma 2.3(2), we have $\mathfrak{N}_\Pi = LF_\sigma(n)$, where n is the least σ-local definition of the formation $\mathfrak{N}_\Pi$ and $n(\sigma_i) = (1)$ for all $\sigma_i \in \Pi$, $n(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Thus, $\mathfrak{H} = \mathfrak{N}_\Pi$ is a σ-local formation of classical type. Let h be the canonical σ-local definition of the formation $\mathfrak{H}$.

By Lemma 2.9, we have $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:

1) $G = P$ is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) = \{1, G\}$;

2) P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$;

3) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is either a monolithic $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with monolith $Q = K^{h(\sigma_i)} \not\subseteq \Phi(K)$, where $\sigma_i \notin \sigma(Q)$, or a minimal non-$h(\sigma_i)$-group of one of the following types: a) the quaternion group of order 8, if $2 \notin \sigma_i$; b) an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$; c) a cyclic q-group, $q \notin \sigma_i$.

If condition 1) holds for G, then, obviously, G satisfies condition (1) of the theorem.

Let condition 2) hold for G. It follows from Lemma 2.3(2) that $h(\sigma_i) = \mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \Pi$ and $h(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. We show that in this case $G = P$ is a simple non-σ-primary τ-minimal non-$\mathfrak{G}_{\sigma_i}$-group for any $\sigma_i \in \sigma(G)$ and $\sigma(G) \subseteq \sigma(\mathfrak{H})$.

Indeed, since $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$, we have $h(\sigma_i) \neq \varnothing$. Therefore, $\sigma(P) \subseteq \sigma(\mathfrak{H})$ by Lemma 2.10(5). On the other hand, since $|\sigma(P)| > 1$, then for $\sigma_i, \sigma_j \in \sigma(P)$, where $i \neq j$, we have

$$G/P \in h(\sigma_i) \cap h(\sigma_j) = \mathfrak{G}_{\sigma_i} \cap \mathfrak{G}_{\sigma_j} = (1).$$

Therefore, and since G is monolithic, we conclude that $G = P$ is a simple non-σ-primary group such that $\sigma(G) \subseteq \sigma(\mathfrak{H})$. Since $\mathfrak{G}_{\sigma_i}$ is a hereditary formation, $\mathfrak{G}_{\sigma_i}$ is a τ-closed formation for any subgroup functor τ. Therefore, by [2, Remark 2.2.12], the $\overline{\tau}$-minimality condition for G can be replaced by the τ-minimality condition. This means that G is a τ-minimal non-$\mathfrak{G}_{\sigma_i}$-group for all $\sigma_i \in \sigma(P)$. Consequently, G satisfies condition (2) of the theorem.

Finally, let condition 3) hold for G. Since $Q = K^{h(\sigma_i)}$, then $\sigma_i \in \sigma(\mathfrak{H})$ and $\Phi(K) = 1$ since $h(\sigma_i) = \mathfrak{G}_{\sigma_i}$ is a saturated formation.

Let us show that K is a simple σ_j-group, $j \neq i$. Indeed, since $K \in \mathfrak{H} \subseteq \mathfrak{N}_\sigma$, it follows that $K = Q$ is a simple σ-primary group due to the monolithicity and σ-nilpotency of K. Consequently, K is a σ_j-group, where $j \neq i$. Moreover, since K is an $\overline{\tau}$-minimal non-$\mathfrak{G}_{\sigma_i}$-group, it follows that $\tau(K) = \{1, K\}$. Therefore, G satisfies condition (3) of the theorem.

Sufficiency. Let $\mathfrak{F}$ be a formation satisfying the conditions of the theorem, h be the canonical σ-local definition of the formation $\mathfrak{H}$. By Lemmas 3.5 and 2.3(2), we have $\mathfrak{H} = \mathfrak{N}_{\sigma(\mathfrak{H})}$ and $h(\sigma_i) = \mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \sigma(\mathfrak{H})$, $h(\sigma_i) = \varnothing$ for all $\sigma_i \notin \sigma(\mathfrak{H})$.

If condition (1) holds for $\mathfrak{F}$, then obviously, by the condition 1) Lemma 2.9, the formation $\mathfrak{F}$ is a minimal τ-closed σ-local non- $\mathfrak{H}$-formation.

Suppose that condition (2) holds for $\mathfrak{F}$. Since $\mathfrak{G}_{\sigma_i}$ is a τ-closed formation, by [2, Remark 2.2.12] G is an $\overline{\tau}$-minimal non-$\mathfrak{G}_{\sigma_i}$-group for any $\sigma_i \in \sigma(G)$. Therefore, G satisfies condition 2) of Lemma 2.9, and hence $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation.

Now let condition (3) hold for the formation $\mathfrak{F}$. We show that in this case, conditions 3) of Lemma 2.9 hold for $\mathfrak{F}$. Indeed, since $h(\sigma_i) = \mathfrak{G}_{\sigma_i}$ and $h(\sigma_j) = \mathfrak{G}_{\sigma_j}$, it follows that $K = K^{h(\sigma_i)}$ is a monolithic τ-minimal non-$h(\sigma_i)$-group, $\sigma_i \notin \sigma(K)$. Moreover, $\Phi(K) = 1$ and $1 = K^{h(\sigma_j)} \subseteq K$. Consequently, conditions 3) of Lemma 2.9 are satisfied for the group G. Therefore, $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation. □

In the case where $\mathfrak{H} = \mathfrak{N}_\sigma$ is the formation of all σ-nilpotent groups, Theorem 4.1 has the following special case.

Theorem 4.2. *If and only if* $\mathfrak{F}$ *is a minimal* τ*-closed* σ*-local non-*σ*-nilpotent formation, then* $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$ *and one of the following conditions holds:*

1) *G is a simple non-σ-primary τ-minimal non-$\mathfrak{G}_{\sigma_i}$-group for any $\sigma_i \in \sigma(G)$;*

2) $G = P \rtimes K$, *where* $P = C_G(P)$ *is a* p*-group,* $p \in \sigma_i$, *and* K *is a simple* σ_j*-group* $(j \neq i)$ *such that* $\tau(K) = \{1, K\}$.

In the case where τ is a trivial subgroup functor, we have

Corollary 4.3 [17, Corollary 2.9]. *If and only if* $\mathfrak{F}$ *is a minimal* σ*-local non-*σ*-nilpotent formation when* $\mathfrak{F} = l_\sigma \mathrm{form}\, G$ *and one of the following conditions holds:*

1) *G is a simple non-σ-primary group;*

2) $G = P \rtimes K$, *where* $P = C_G(P)$ *is a* p*-group,* $p \in \sigma_i$, *and* K *is a simple* σ_j*-group,* $j \neq i$.

In particular, if $\sigma = \sigma^1 = \{\{2\}, \{3\}, \{5\}, \ldots\}$, from Theorem 4.1 we obtain

Corollary 4.4. *Let* $\mathfrak{F}$ *and* $\mathfrak{H}$ *be* τ*-closed local formations such that* $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}$. *If and only if* $\mathfrak{F}$ *is a minimal* τ*-closed local non-*$\mathfrak{H}$*-formation when* $\mathfrak{F} = \tau^l \mathrm{form}\, G$, *where* G *is one of the following groups:*

(1) *a group of prime order* $p \notin \pi(\mathfrak{H})$;

(2) *a simple non-abelian* τ*-minimal non-*$\mathfrak{N}_p$*-group for any* $p \in \pi(G) \subseteq \pi(\mathfrak{H})$;

(3) *a Schmidt group,* $\pi(G) \subseteq \pi(\mathfrak{H})$.

If τ is the trivial subgroup functor, then we have

Corollary 4.5. *Let* $\mathfrak{F}$ *and* $\mathfrak{H}$ *be local formations such that* $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}$. *If and only if* $\mathfrak{F}$ *is a minimal local non-*$\mathfrak{H}$*-formation, then* $\mathfrak{F} = l\mathrm{form}\, G$, *where* G *is one of the following groups:*

(1) *a group of prime order* $p \notin \pi(\mathfrak{H})$;

(2) *a simple non-abelian group,* $\pi(G) \subseteq \pi(\mathfrak{H})$;

(3) *a Schmidt group,* $\pi(G) \subseteq \pi(\mathfrak{H})$.

Furthermore, if $\mathfrak{H} = \mathfrak{N}$ is the formation of all nilpotent groups from Theorem 4.1 we obtain the following well-known result.

Corollary 4.6 [2, Corollary 2.4.4]. *If and only if* $\mathfrak{F}$ *is a minimal* τ*-closed local non-nilpotent formation when* $\mathfrak{F} = \tau^l \mathrm{form}\, G$ *where* G *is either a simple non-abelian* τ*-minimal non-* $\mathfrak{N}_p$*-group for any* $p \in \pi(G)$, *or a Schmidt group.*

If τ is a trivial subgroup functor, then we have

Corollary 4.7 [1, Corollary 19.10]. *If and only if* $\mathfrak{F}$ *is a minimal local non-nilpotent formation, then* $\mathfrak{F} = l\mathrm{form}\, G$ *and one of the following conditions holds:*

(1) *G is a Schmidt group;*

(2) *G is a simple non-abelian group.*

The question of the existence of $\mathfrak{H}_\sigma^\tau$-critical formations, in the case where $\mathfrak{H} \subseteq \mathfrak{N}_\sigma$, is decided by

Theorem 4.8. *Let* $\mathfrak{F}$ *and* $\mathfrak{H}$ *be* τ*-closed* σ*-local formations such that* $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$ *Then* $\mathfrak{F}$ *contains at least one minimal* τ*-closed* σ*-local non-*$\mathfrak{H}$*-subformation.*

Proof. Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed σ-local formations from the hypothesis of the theorem. If there exists a σ_k such that $\sigma_k \in \sigma(\mathfrak{F}) \setminus \sigma(\mathfrak{H})$, then by Lemma 2.10 we have $\mathfrak{G}_{\sigma_k} \subseteq \mathfrak{F}$ and $\mathfrak{G}_{\sigma_k} \nsubseteq \mathfrak{H}$. Since $\mathfrak{G}_{\sigma_k}$ is a τ-closed σ-local formation, and its only proper σ-local subformation is $(1) \subseteq \mathfrak{H}$, then $\mathfrak{G}_{\sigma_k}$ is the desired $\mathfrak{H}_\sigma^\tau$ is a critical formation from $\mathfrak{F}$.

In what follows, we will assume that $\sigma(\mathfrak{F}) \subseteq \sigma(\mathfrak{H})$.

By Lemma 3.5, we have $\mathfrak{H} = \mathfrak{N}_{\sigma(\mathfrak{H})}$. Let h be the canonical σ-local definition of $\mathfrak{H}$. By Lemma 2.3(2), we have $h(\sigma_i) = \mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \sigma(\mathfrak{H})$ and $h(\sigma_i) = \varnothing$ for all $\sigma_i \notin \sigma(\mathfrak{H})$. Since $\mathfrak{F} \nsubseteq \mathfrak{H}$, by Lemma 2.11 there exists at least one $\sigma_i \in \sigma(\mathfrak{F})$ such that $f(\sigma_i) \nsubseteq h(\sigma_i) = \mathfrak{G}_{\sigma_i}$. We choose a group K_i of minimal order in $f(\sigma_i) \setminus \mathfrak{G}_{\sigma_i}$. Since the formation $\mathfrak{G}_{\sigma_i}$ is τ-closed, K_i is a monolithic τ-minimal non-$\mathfrak{G}_{\sigma_i}$-group with monolith $Q_i = K_i^{\mathfrak{G}_{\sigma_i}}$. Among all such groups K_i, we choose a group K_j of smallest order. Let $K := K_j$. Then K is a monolithic τ-minimal non-$\mathfrak{G}_{\sigma_j}$-group with monolith $Q = K^{\mathfrak{G}_{\sigma_j}}$.

Assume that Q is not a σ-primary group and let $\sigma_k \in \sigma(Q) \setminus \{\sigma_j\}$. Then $O_{\sigma_k, \sigma_k'}(K) = 1$ due to the monolithicity of K. Since $K \in \mathfrak{F}$, then $K \simeq K/O_{\sigma_j, \sigma_j'}(K) \in f(\sigma_j)$. It is clear that $K \notin \mathfrak{G}_{\sigma_k} = h(\sigma_k)$. Therefore, $K \in f(\sigma_k) \setminus h(\sigma_k)$. If, in addition, $K/Q \notin h(\sigma_k)$, then $K/Q \in f(\sigma_k) \setminus h(\sigma_k)$. The latter contradicts the choice of the group K, since $|K/Q| < |K|$. Therefore, $K/Q \in h(\sigma_k) = \mathfrak{G}_{\sigma_k}$. Thus, $K/Q \in \mathfrak{G}_{\sigma_j} \cap \mathfrak{G}_{\sigma_k} = (1)$. Consequently, K is a simple non-σ-primary τ-minimal non-$\mathfrak{G}_{\sigma_j}$-group.

Now let $H \in \tau(K) \setminus \{K\}$. Then $H \in f(\sigma_k)$, since $K \in f(\sigma_k)$ and $f(\sigma_k)$ is a τ-closed formation. Suppose that $H \notin h(\sigma_k)$. Then since $|H| < |K|$, we obtain a contradiction with the choice of $|K|$. Thus, K is a simple non-σ-primary τ-minimal non-$\mathfrak{G}_{\sigma_k}$-group for any $\sigma_k \in \sigma(K)$. Thus, the group K satisfies condition (2) of Theorem 4.1. Therefore, $\mathfrak{L} = l_\sigma^\tau \mathrm{form}\, K$ is the desired $\mathfrak{H}_\sigma^\tau$-critical formation from $\mathfrak{F}$.

Now let Q be a σ-primary group, i. e., a σ_k-group for some $k \neq j$. Then $K \neq Q$, since $\sigma(\mathfrak{F}) \subseteq \subseteq \sigma(\mathfrak{H})$. Moreover, $Q = O_{\sigma_k,\sigma_k'}(K) = O_{\sigma_k}(K)$ since Q is a monolith of K and $k \neq j$. Since $K \in \mathfrak{F}$, then $K/Q = K/O_{\sigma_k,\sigma_k'}(K) \in f(\sigma_k)$. Therefore, $K/Q \in f(\sigma_k) \cap \mathfrak{G}_{\sigma_j} \neq \varnothing$. Let A be a group of minimal order in $f(\sigma_k) \cap \mathfrak{G}_{\sigma_j}$. Then A is a simple σ_j-group and $\tau(A) = \{1, A\}$.

Let $p \in \sigma_k$. Since $O_p(A) = 1$, by Lemma 2.12 there exists a faithful irreducible F_pA-module P, where F_p is a field of p elements. Let $G = P \rtimes A$. Then $P = C_G(P)$ and the group G satisfies condition (3) of Theorem 4.1. Consequently, $\mathfrak{L} = l_\sigma^\tau \operatorname{form} G$ is the desired $\mathfrak{H}_\sigma^\tau$-critical formation from $\mathfrak{F}$. □

In particular, if τ is the trivial subgroup functor from Theorem 4.8, we obtain

Corollary 4.9 [22, Theorem 3.8]. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be σ-local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$. Then $\mathfrak{F}$ has at least one minimal σ-local non-$\mathfrak{H}$-subformation.*

If $\mathfrak{H} = \mathfrak{N}_\sigma$ is the formation of all σ-nilpotent groups, then we obtain the following important special case of Theorem 4.8.

Theorem 4.10. *Let $\mathfrak{F}$ be a non-σ-nilpotent τ-closed σ-local formation. Then $\mathfrak{F}$ has at least one minimal τ-closed σ-local non-σ-nilpotent subformation.*

If τ is a trivial subgroup functor, we obtain

Corollary 4.11 [22, Corollary 3.9]. *Let $\mathfrak{F}$ be a non-σ-nilpotent σ-local formation. Then $\mathfrak{F}$ has at least one minimal σ-local non-σ-nilpotent subformation.*

Recall that if $\mathfrak{M}$ and $\mathfrak{H}$ are formations such that $\mathfrak{M} \subseteq \mathfrak{H}$. Then the formation $\mathfrak{M}$ is called a subformation of $\mathfrak{H}$ or, alternatively, an $\mathfrak{H}$-subformation.

Theorem 4.12. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be l_σ^τ-formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$. Then and only if the $\mathfrak{H}_\sigma^\tau$-defect of $\mathfrak{F}$ is* 1, *when $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{L}$, where $\mathfrak{M}$ is a τ-closed σ-local subformation of $\mathfrak{H}$, $\mathfrak{L}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation, such that:*

(1) *every τ-closed $\mathfrak{H}$-subformation of $\mathfrak{F}$ is contained in $\mathfrak{M} \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{H})$;*

(2) *every l_σ^τ-formation $\mathfrak{X}$ from $\mathfrak{F}$ such that $\mathfrak{X} \not\subseteq \mathfrak{H}$, has the form $\mathfrak{L} \vee_\sigma^\tau (\mathfrak{X} \cap \mathfrak{H})$.*

Proof. Necessity. Let $\mathfrak{F}$ be a τ-closed σ-local formation with $\mathfrak{H}_\sigma^\tau$-defect 1. Since $\mathfrak{F} \not\subseteq \mathfrak{H}$, then by Theorem 4.8 $\mathfrak{F}$ contains some minimal τ-closed σ-local not $\mathfrak{H}$-formation $\mathfrak{L}$. By the hypothesis of the theorem, $\mathfrak{M} = \mathfrak{F} \cap \mathfrak{H}$ is the maximal l_σ^τ-subformation of $\mathfrak{F}$. Therefore, $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{L}$.

Sufficiency. Let $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{L}$, where $\mathfrak{L}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation, and $\mathfrak{M}$ is an l_σ^τ-formation of $\mathfrak{H}$. Then, by Lemma 3.3, $\mathfrak{H}_\sigma^\tau$-defect of $\mathfrak{F}$ is equal to 1.

We now show that statements (1) and (2) hold. Since $\mathfrak{L} \cap \mathfrak{H}$ is a maximal τ-closed σ-local subformation of $\mathfrak{L}$, it follows from Lemmas 2.1 and 2.13 of the lattice isomorphism

$$\mathfrak{F}/_\sigma^\tau(\mathfrak{M} \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{H})) = (\mathfrak{M} \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{H}) \vee_\sigma^\tau \mathfrak{L})/_\sigma^\tau(\mathfrak{M} \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{H})) \simeq$$

$$\simeq \mathfrak{L}/_\sigma^\tau(\mathfrak{L} \cap ((\mathfrak{L} \cap \mathfrak{H}) \vee_\sigma^\tau \mathfrak{M})) = \mathfrak{L}/_\sigma^\tau((\mathfrak{L} \cap \mathfrak{H}) \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{M})) = \mathfrak{L}/_\sigma^\tau \mathfrak{L} \cap \mathfrak{H}$$

we get that $\mathfrak{M} \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{H})$ is the maximal l_σ^τ-subformation of $\mathfrak{F}$. Since $\mathfrak{F} \not\subseteq \mathfrak{H}$, then every $\mathfrak{H}$-subformation of $\mathfrak{F}$ is included in $(\mathfrak{L} \cap \mathfrak{H}) \vee_\sigma^\tau \mathfrak{M}$. Therefore, assertion (1) holds.

Let us now show that in $\mathfrak{F}$ there are no minimal τ-closed σ-local non-$\mathfrak{H}$-formations different from $\mathfrak{L}$. Suppose that this is false, and let $\mathfrak{L}_1$ be the minimal τ-closed σ-local non-$\mathfrak{H}$-formation in $\mathfrak{F}$ such that $\mathfrak{L}_1 \neq \mathfrak{L}$. Since the $\mathfrak{H}_\sigma^\tau$-defects of $\mathfrak{L}$ and $\mathfrak{L}_1$ are equal to 1 and $\mathfrak{L} \cap \mathfrak{L}_1 \subseteq \mathfrak{H}$, by Lemma 2.9 we have

$$|\mathfrak{L} \vee_\sigma^\tau \mathfrak{L}_1 : \mathfrak{H} \cap (\mathfrak{L} \vee_\sigma^\tau \mathfrak{L}_1)|_\sigma^\tau = |\mathfrak{L} : \mathfrak{H} \cap \mathfrak{L}|_\sigma^\tau + |\mathfrak{L}_1 : \mathfrak{H} \cap \mathfrak{L}_1|_\sigma^\tau - |\mathfrak{L} \cap \mathfrak{L}_1 : \mathfrak{H} \cap (\mathfrak{L} \cap \mathfrak{L}_1)|_\sigma^\tau = 2.$$

The latter contradicts Lemma 3.2, since $\mathfrak{L} \vee_\sigma^\tau \mathfrak{L}_1 \subseteq \mathfrak{F}$. Thus, in the formation $\mathfrak{F}$ there are no minimal τ-closed σ-local non-$\mathfrak{H}$-formations distinct from $\mathfrak{L}$.

Now let $\mathfrak{X}$ be an arbitrary l_σ^τ-subformation of $\mathfrak{F}$ such that $\mathfrak{X} \not\subseteq \mathfrak{H}$. Then, by what was proved above and Theorem 4.8, we conclude that $\mathfrak{L} \subseteq \mathfrak{X}$. Since $\mathfrak{X}$ has $\mathfrak{H}_\sigma^\tau$-defect equal to 1, $\mathfrak{X} \cap \mathfrak{H}$ is the maximal τ-closed σ-local subformation of $\mathfrak{X}$. Therefore, $\mathfrak{X} = \mathfrak{L} \vee_\sigma^\tau (\mathfrak{X} \cap \mathfrak{H})$, i. e., assertion (2) holds. □

In the case when $\mathfrak{H} = \mathfrak{N}_\sigma$, from Theorem 4.12 we obtain the following result.

Theorem 4.13. *Let $\mathfrak{F}$ be a τ-closed σ-local non-σ-nilpotent formation. If and only if the σ-nilpotent l_σ^τ-defect of a formation $\mathfrak{F}$ is* 1 *when $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{L}$, where $\mathfrak{M}$ is a σ-nilpotent τ-closed σ-local subformation of $\mathfrak{F}$, $\mathfrak{L}$ is a minimal τ-closed σ-local non-σ-nilpotent formation, and:*

(1) *every σ-nilpotent τ-closed subformation of $\mathfrak{F}$ is included in $\mathfrak{M} \vee_\sigma^\tau (\mathfrak{L} \cap \mathfrak{N}_\sigma)$;*

(2) *every non-σ-nilpotent l_σ^τ-subformation $\mathfrak{X}$ of $\mathfrak{F}$ has the form $\mathfrak{L} \vee_\sigma^\tau (\mathfrak{X} \cap \mathfrak{N}_\sigma)$.*

In the case where $\tau = s$ is the identity subgroup functor, Theorem 4.12 implies

Corollary 4.14. *Let $\mathfrak{F}$ be a hereditary σ-local non-σ-nilpotent formation. If and only if the σ-nilpotent l_σ^s-defect of $\mathfrak{F}$ is 1 when $\mathfrak{F} = \mathfrak{M} \vee_\sigma^s \mathfrak{L}$, where $\mathfrak{M}$ is a σ-nilpotent hereditary σ-local subformation of $\mathfrak{F}$, $\mathfrak{L}$ is a minimal hereditary σ-local non-σ-nilpotent formation, and:*

(1) *every σ-nilpotent hereditary subformation of $\mathfrak{F}$ is included in $\mathfrak{M} \vee_\sigma^s (\mathfrak{L} \cap \mathfrak{N}_\sigma)$;*

(2) *every non-σ-nilpotent l_σ^s-subformation $\mathfrak{X}$ of $\mathfrak{F}$ has the form $\mathfrak{L} \vee_\sigma^s (\mathfrak{X} \cap \mathfrak{N}_\sigma)$.*

If $\tau(G) = s_n(G)$ is the set of all normal subgroups of G for any group G, then we obtain

Corollary 4.15. *Let $\mathfrak{F}$ be a normally hereditary σ-local non- σ-nilpotent formation. If and only if the σ-nilpotent $l_\sigma^{s_n}$-defect of $\mathfrak{F}$ is 1 when $\mathfrak{F} = \mathfrak{M} \vee_\sigma^{s_n} \mathfrak{L}$, where $\mathfrak{M}$ is a σ-nilpotent σ-local subformation of $\mathfrak{F}$, $\mathfrak{L}$ is a minimal normally hereditary σ-local non-σ-nilpotent formation, and:*

(1) *every σ-nilpotent $l_\sigma^{s_n}$-subformation of $\mathfrak{F}$ is in the $\mathfrak{M} \vee_\sigma^{s_n} (\mathfrak{L} \cap \mathfrak{N}_\sigma)$;*

(2) *every non-σ-nilpotent $l_\sigma^{s_n}$-subformation of $\mathfrak{X}$ of $\mathfrak{F}$ has the form $\mathfrak{L} \vee_\sigma^{s_n} (\mathfrak{X} \cap \mathfrak{N}_\sigma)$.*

In particular, if $\sigma = \sigma^1 = \{\{2\},\{3\},\{5\},\ldots\}$ from Theorem 4.12 we obtain

Corollary 4.16. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}$. If and only if the $\mathfrak{H}_l^\tau$-defect of the formation $\mathfrak{F}$ is 1, when $\mathfrak{F} = \mathfrak{M} \vee_l^\tau \mathfrak{L}$, where $\mathfrak{M}$ is a τ-closed local subformation of $\mathfrak{H}$, $\mathfrak{L}$ is the minimal τ-closed local not $\mathfrak{H}$-formation, for In this case:*

(1) *every τ-closed $\mathfrak{H}$-subformation of $\mathfrak{F}$ is contained in $\mathfrak{M} \vee_l^\tau (\mathfrak{L} \cap \mathfrak{H})$;*

(2) *every τ-closed local subformation $\mathfrak{X}$ of $\mathfrak{F}$ such that $\mathfrak{X} \not\subseteq \mathfrak{H}$ has the form $\mathfrak{L} \vee_l^\tau (\mathfrak{X} \cap \mathfrak{H})$.*

If τ is a trivial subgroup functor, then

Corollary 4.17. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}$. If and only if the $\mathfrak{H}_l$-defect of the formation $\mathfrak{F}$ is 1, when $\mathfrak{F} = \mathfrak{M} \vee_l \mathfrak{L}$, where $\mathfrak{M}$ is a local subformation of $\mathfrak{H}$, $\mathfrak{L}$ is the minimal local not $\mathfrak{H}$ is a formation, and:*

(1) *every $\mathfrak{H}$-subformation of $\mathfrak{F}$ is contained in $\mathfrak{M} \vee_l (\mathfrak{L} \cap \mathfrak{H})$;*

(2) *every local subformation $\mathfrak{X}$ of $\mathfrak{F}$ such that $\mathfrak{X} \not\subseteq \mathfrak{H}$ has the form $\mathfrak{L} \vee_l (\mathfrak{X} \cap \mathfrak{H})$.*

Furthermore, if $\mathfrak{H} = \mathfrak{N}$ is the formation of all nilpotent groups from Theorem 4.12, we obtain the following well-known result.

Corollary 4.18 [1, Lemma 20.5]. *Then precisely the nilpotent defect of a local formation $\mathfrak{F}$ is equal to 1 when $\mathfrak{F} = \mathfrak{M} \vee_l \mathfrak{L}$, where $\mathfrak{M}$ is a nilpotent local formation, $\mathfrak{L}$ is a minimal local non-nilpotent formation, and:*

(1) *every nilpotent subformation of $\mathfrak{F}$ is contained in $\mathfrak{M} \vee_l (\mathfrak{L} \cap \mathfrak{N})$;*

(2) *every non-nilpotent local subformation $\mathfrak{X}$ of $\mathfrak{F}$ has the form $\mathfrak{L} \vee_l (\mathfrak{X} \cap \mathfrak{N})$.*

Theorem 4.19. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed σ-local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$. Then if $\sigma(\mathfrak{F}) \subseteq \sigma(\mathfrak{H})$, then the following conditions are equivalent:*

(1) $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|_\sigma^\tau = 1$;

(2) *in $\mathfrak{F}$ each of its τ-closed σ-local non-$\mathfrak{H}$-subformation is complemented;*

(3) *in $\mathfrak{F}$ each of its τ-closed σ-local subformations $\mathfrak{M}$ with $|\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|_\sigma^\tau = 1$ is complemented.*

Proof. Let (1) hold and $\mathfrak{M}$ is a τ-closed σ-local subformation of $\mathfrak{F}$. Then if $\mathfrak{M} \not\subseteq \mathfrak{H}$, then by Theorem 4.12 we have $\mathfrak{M} = \mathfrak{L} \vee_\sigma^\tau (\mathfrak{M} \cap \mathfrak{H})$, where $\mathfrak{L}$ is the minimal τ-closed σ-local not $\mathfrak{H}$-formation. Let $\Pi = \sigma(\mathfrak{F})$, $\Pi_1 = \sigma(\mathfrak{M})$ and $\Pi_2 = \Pi \setminus \Pi_1$. We show that $\mathfrak{N}_{\Pi_2}$ is the complement of $\mathfrak{M}$ in $\mathfrak{F}$. It is clear that $\mathfrak{N}_{\Pi_2} \cap \mathfrak{M} = (1)$. We show that $\mathrm{form}(\mathfrak{M} \cup \mathfrak{N}_{\Pi_2}) = \mathfrak{F}$.

By Theorem 4.12 we have $\mathfrak{F} = \mathfrak{L} \vee_\sigma^\tau \mathfrak{M}_1$, where $\mathfrak{M}_1 \subseteq \mathfrak{H}$. On the other hand,

$$\mathfrak{M} = \mathfrak{L} \vee_\sigma^\tau (\mathfrak{M} \cap \mathfrak{H}) = \mathfrak{L} \vee_\sigma^\tau \mathfrak{N}_{\Pi_1},$$

because $\mathfrak{M} \cap \mathfrak{H} \subseteq \mathfrak{M} \cap \mathfrak{N}_\sigma = \mathfrak{N}_{\Pi_1}$. Now in force Lemmas 2.14 and 2.5 we have

$$\mathrm{form}(\mathfrak{M} \cup \mathfrak{N}_{\Pi_2}) = \mathfrak{M} \oplus \mathfrak{N}_{\Pi_2} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{N}_{\Pi_2} = (\mathfrak{L} \vee_\sigma^\tau \mathfrak{N}_{\Pi_1}) \vee_\sigma^\tau \mathfrak{N}_{\Pi_2} = \mathfrak{L} \vee_\sigma^\tau \mathfrak{N}_{\Pi} = \mathfrak{F}.$$

Thus, the formation $\mathfrak{N}_{\Pi_2}$ is the complement of $\mathfrak{M}$ in $\mathfrak{F}$.

Clearly, if assertion (2) holds, then assertion (3) holds, since any τ-closed σ-local subformation of $\mathfrak{M}$ with $|\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|_\sigma^\tau = 1$ is not an $\mathfrak{H}$-subformation of $\mathfrak{F}$.

Now let (3) hold. We will show that condition (1) is satisfied. By the hypothesis of the theorem, $\mathfrak{F} \not\subseteq \mathfrak{H}$. Therefore, by Lemma 2.4, $\mathfrak{F}$ has a minimal τ-closed σ-local non-$\mathfrak{H}$-formation $\mathfrak{L}$. Let $\mathfrak{M} = \mathfrak{H} \cap \mathfrak{F}$ and $\mathfrak{F}_1 = \mathfrak{M} \vee_\sigma^\tau \mathfrak{L}$. By Theorem 4.12, we have $|\mathfrak{F}_1 : \mathfrak{H} \cap \mathfrak{F}|_\sigma^\tau = 1$.

Therefore, by the hypothesis of the theorem, $\mathfrak{F}$ contains a subformation $\mathfrak{M}_1$ such that $\mathfrak{M}_1 \cap \mathfrak{F}_1 = (1)$ and $\mathfrak{F} = \mathrm{form}(\mathfrak{F}_1 \cup \mathfrak{M}_1)$. Now applying Lemmas 2.14 and 2.5, we obtain that $\mathfrak{F} = \mathfrak{F}_1 \oplus \mathfrak{M}_1$ and the formation $\mathfrak{M}_1$ τ-closed and σ-local.

Suppose that $\mathfrak{M}_1 \neq (1)$. Then if $\sigma_i \in \sigma(\mathfrak{M}_1)$, then $\sigma_i \in \sigma(\mathfrak{F}) \subseteq \sigma(\mathfrak{H})$ by the hypothesis of the theorem. Therefore, by Lemma 2.4(ii), the inclusions hold

$$\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{M}_1 \cap (\mathfrak{F} \cap \mathfrak{H}) \subseteq \mathfrak{M}_1 \cap \mathfrak{F}_1 = (1).$$

The resulting contradiction shows that $\mathfrak{M}_1 = (1)$. Therefore, $\mathfrak{F} = \mathfrak{F}_1 \oplus \mathfrak{M}_1 = \mathfrak{F}_1$. Therefore, $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|^{\tau}_{\sigma} = 1$. □

Remark 4.20. *The condition $\sigma(\mathfrak{F}) \subseteq \sigma(\mathfrak{H})$ in Theorem 4.19 cannot be omitted, since the presence of a complement in $\mathfrak{F}$ for each of its τ-closed σ-local non- $\mathfrak{H}$-subformations, as well as the presence of a complement in $\mathfrak{F}$ for each τ-closed σ-local subformation $\mathfrak{M}$ of $\mathfrak{F}$ with $|\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|^{\tau}_{\sigma} = 1$ does not imply the equality $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|^{\tau}_{\sigma} = 1$. Indeed, let $\mathfrak{H} = \mathfrak{G}_{\sigma_i}$ and $\mathfrak{F} = \mathfrak{H} \vee^{\tau}_{\sigma} \mathfrak{G}_{\sigma_j} \vee^{\tau}_{\sigma} \mathfrak{G}_{\sigma_k}$, where $\sigma_j, \sigma_k \in \sigma \setminus \{\sigma_i\}$, $j \neq k$. Then, by Lemmas 2.5 and 2.14 we have $\mathfrak{F} = \mathfrak{H} \oplus \mathfrak{G}_{\sigma_j} \oplus \mathfrak{G}_{\sigma_k}$. By Theorem 4.1 and Lemma 3.3 we have $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|^{\tau}_{\sigma} = 2$. However, as is easy to see, every τ-closed σ-local non-$\mathfrak{H}$-subformation of $\mathfrak{F}$, as well as every τ-closed σ-local subformation of $\mathfrak{F}$ with $\mathfrak{H}^{\tau}_{\sigma}$-defect 1, have complement in $\mathfrak{F}$.*

However, the following holds:

Corollary 4.21. *Let $\mathfrak{F}$ be a τ-closed σ-local non-σ-nilpotent formation. Then the following conditions are equivalent:*

(1) $|\mathfrak{F} : \mathfrak{N}_{\sigma} \cap \mathfrak{F}|^{\tau}_{\sigma} = 1$;

(2) *in $\mathfrak{F}$ each of its τ-closed σ-local non-σ-nilpotent subformations is complemented;*

(3) *in $\mathfrak{F}$ each of its τ-closed σ-local subformations $\mathfrak{M}$ with $|\mathfrak{M} : \mathfrak{N}_{\sigma} \cap \mathfrak{M}|^{\tau}_{\sigma} = 1$ is complemented.*

In particular, if $\sigma = \sigma^1 = \{\{2\}, \{3\}, \{5\}, \ldots\}$, from Theorem 4.19 we have

Corollary 4.22. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be such local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}$. Then if $\pi(\mathfrak{F}) \subseteq \pi(\mathfrak{H})$, then the following conditions are equivalent:*

(1) $|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|_l = 1$;

(2) *in $\mathfrak{F}$ each of its local non- $\mathfrak{H}$-subformations is complemented;*

(3) *in $\mathfrak{F}$ each of its local subformations $\mathfrak{M}$ with $|\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|_l = 1$ is complemented.*

Moreover, if $\mathfrak{H} = \mathfrak{N}$, from Theorem 4.19 we obtain the following well-known result.

Corollary 4.23 [2, Corollary 5.2.12]. *Let $\mathfrak{F}$ be a non-nilpotent τ-closed local formation. Then the following conditions are equivalent:*

(1) $|\mathfrak{F} : \mathfrak{N} \cap \mathfrak{F}|^{\tau}_{l} = 1$;

(2) *In $\mathfrak{F}$ every non-nilpotent τ-closed local subformation is τ-complemented;*

(3) *In $\mathfrak{F}$ every τ-closed local subformation $\mathfrak{M}$ with $|\mathfrak{M} : \mathfrak{N} \cap \mathfrak{M}|^{\tau}_{l} = 1$ is τ-complemented.*

5. Reducible l^{τ}_{σ}-formations of bounded $\mathfrak{H}^{\tau}_{\sigma}$-defect

The main result of this section is the following theorem, which describes reducible τ-closed σ-local formations of finite $\mathfrak{H}^{\tau}_{\sigma}$-defect.

Theorem 5.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed σ-local formations such that $\mathfrak{F} \not\subseteq \mathfrak{H} \subseteq \mathfrak{N}_{\sigma}$ and let $\mathfrak{F}$ be l^{τ}_{σ}-reducible. If and only if $\mathfrak{H}^{\tau}_{\sigma}$-defect of formation $\mathfrak{F}$ is equal to k, when $\mathfrak{F}$ satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L} \vee^{\tau}_{\sigma} \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation $\mathfrak{H}^{\tau}_{\sigma}$-defect t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is such a τ-closed σ-local formation $\mathfrak{H}^{\tau}_{\sigma}$-defect $k-1$, such that $\mathfrak{L} \cap \mathfrak{M}$ is the maximal τ-closed σ-local subformation of the formation $\mathfrak{L}$;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^{\tau}_{\sigma} \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation $\mathfrak{H}^{\tau}_{\sigma}$-defect k, $\mathfrak{M}$ is such τ-closed σ-local formation such that $\mathfrak{M} \subseteq \mathfrak{H}$ and $\mathfrak{M} \not\subseteq \mathfrak{L}$.*

Proof. Sufficiency. Let $\mathfrak{F}$ satisfy condition (1). Since $\mathfrak{L} \cap \mathfrak{M}$ is the unique maximal τ-closed σ-local subformation of $\mathfrak{L}$, it follows that $|\mathfrak{L} \cap \mathfrak{M} : \mathfrak{H} \cap (\mathfrak{L} \cap \mathfrak{M})|^{\tau}_{\sigma} = t-1$. Therefore, by Lemma 3.3 we have

$$|\mathfrak{F} : \mathfrak{H} \cap \mathfrak{F}|^{\tau}_{\sigma} = |\mathfrak{L} : \mathfrak{H} \cap \mathfrak{L}|^{\tau}_{\sigma} + |\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|^{\tau}_{\sigma} - |\mathfrak{L} \cap \mathfrak{M} : \mathfrak{H} \cap (\mathfrak{L} \cap \mathfrak{M})|^{\tau}_{\sigma} = t + k - 1 - (t-1) = k.$$

Now let the formation $\mathfrak{F}$ satisfy condition (2). Then by Lemma 3.3 we get

$$|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_{\sigma}^{\tau}=|\mathfrak{L}:\mathfrak{H}\cap\mathfrak{L}|_{\sigma}^{\tau}+|\mathfrak{M}:\mathfrak{H}\cap\mathfrak{M}|_{\sigma}^{\tau}-|\mathfrak{L}\cap\mathfrak{M}:\mathfrak{H}\cap(\mathfrak{L}\cap\mathfrak{M})|_{\sigma}^{\tau}=k+0-0=k.$$

Thus, we have $|\mathfrak{F}:\mathfrak{H}\cap\mathfrak{F}|_{\sigma}^{\tau}=k$.

Necessity. We prove the necessity by induction on k. Let $k=1$ and $\mathfrak{F}$ be a τ-closed σ-local formation with $\mathfrak{H}_{\sigma}^{\tau}$-defect 1. Since $\mathfrak{F}\not\subseteq\mathfrak{H}$, then by Theorem 4.8 $\mathfrak{F}$ contains some minimal τ-closed σ-local not $\mathfrak{H}$ is a subformation of $\mathfrak{L}$. Since the $\mathfrak{H}_{\sigma}^{\tau}$-defect of $\mathfrak{F}$ is 1, $\mathfrak{M}=\mathfrak{F}\cap\mathfrak{H}$ is the maximal τ-closed σ-local subformation of $\mathfrak{F}$. Therefore, $\mathfrak{F}=\mathfrak{L}\vee_{\sigma}^{\tau}\mathfrak{M}$ and the formation $\mathfrak{F}$ satisfies condition (2) of the theorem.

Let $k>1$ and assume that the theorem holds for $k-1$. Let $\mathfrak{M}$ denote the maximal τ-closed σ-local subformation of $\mathfrak{F}$ whose $\mathfrak{H}_{\sigma}^{\tau}$-deficit is $k-1$.

Suppose that in $\mathfrak{F}$ there exists an irreducible τ-closed σ-local subformation $\mathfrak{X}$ such that $\mathfrak{X}\not\subseteq\mathfrak{M}$ and $1\leqslant|\mathfrak{X}:\mathfrak{H}\cap\mathfrak{X}|_{\sigma}^{\tau}\leqslant k-1$. Then $\mathfrak{F}=\mathfrak{M}\vee_{\sigma}^{\tau}\mathfrak{X}$. Let $t=|\mathfrak{X}:\mathfrak{H}\cap\mathfrak{X}|_{\sigma}^{\tau}$. If $t=1$, then $\mathfrak{X}\cap\mathfrak{H}$ is the unique maximal τ-closed σ-local subformation of $\mathfrak{X}$. Since $\mathfrak{M}$ is maximal, we have $\mathfrak{M}\cap\mathfrak{H}=\mathfrak{F}\cap\mathfrak{H}$. Therefore, $\mathfrak{X}\cap\mathfrak{H}=\mathfrak{M}\cap\mathfrak{H}$. Therefore, $\mathfrak{X}\cap\mathfrak{M}=\mathfrak{X}\cap\mathfrak{H}$ and the formation $\mathfrak{F}$ satisfies condition (1) of the theorem.

Now let $2\leqslant t\leqslant k-1$ and let any irreducible τ-closed σ-local subformation of $\mathfrak{F}$ with $\mathfrak{H}_{\sigma}^{\tau}$-defect less than t be contained in $\mathfrak{M}$. Let $\mathfrak{X}_1$ be a maximal τ-closed σ-local subformation of $\mathfrak{X}$ such that $|\mathfrak{X}_1:\mathfrak{H}\cap\mathfrak{X}_1|_{\sigma}^{\tau}=t-1$. If $\mathfrak{X}_1$ is l_{σ}^{τ}-irreducible, then by assumption $\mathfrak{X}_1\subseteq\mathfrak{M}$. Therefore, $\mathfrak{X}\cap\mathfrak{M}=\mathfrak{X}_1$ and $\mathfrak{F}$ satisfies condition (1) of the theorem.

Let $\mathfrak{X}_1$ be a reducible τ-closed σ-local formation. Since $t-1<k-1$, then by the induction hypothesis for the formation $\mathfrak{X}_1$ the theorem is true. Therefore, the formation $\mathfrak{X}_1$ satisfies one of the following conditions:

(a) $\mathfrak{X}_1=\mathfrak{L}_1\vee_{\sigma}^{\tau}\mathfrak{M}_1$, where $\mathfrak{L}_1$ is an irreducible τ-closed σ-local formation and $|\mathfrak{L}_1:\mathfrak{H}\cap\mathfrak{L}_1|_{\sigma}^{\tau}=s$, $1\leqslant s\leqslant k-2$, and $\mathfrak{M}_1$ is a τ-closed σ-local formation such that $|\mathfrak{M}_1:\mathfrak{H}\cap\mathfrak{M}_1|_{\sigma}^{\tau}=k-2$ and $\mathfrak{L}_1\cap\mathfrak{M}_1$ is the maximal τ-closed σ-local subformation of the formation $\mathfrak{L}_1$;

(b) $\mathfrak{X}_1=\mathfrak{L}_1\vee_{\sigma}^{\tau}\mathfrak{M}_1$, where $\mathfrak{M}_1\subseteq\mathfrak{H}$, and $\mathfrak{L}_1$ is an irreducible τ-closed σ-local formation such that $|\mathfrak{L}_1:\mathfrak{H}\cap\mathfrak{L}_1|_{\sigma}^{\tau}=k-1$ and $\mathfrak{M}_1\not\subseteq\mathfrak{L}_1$.

Let (b) hold. Then, by assumption, $\mathfrak{L}_1\subseteq\mathfrak{M}$. Moreover, since $\mathfrak{M}\cap\mathfrak{H}=\mathfrak{F}\cap\mathfrak{H}$, we have $\mathfrak{M}_1\subseteq\mathfrak{M}$ and $\mathfrak{X}_1=\mathfrak{L}_1\vee_{\sigma}^{\tau}\mathfrak{M}_1\subseteq\mathfrak{M}$. Consequently, $\mathfrak{X}\cap\mathfrak{M}=\mathfrak{X}_1$ and the formation $\mathfrak{F}$ satisfies condition (1) of the theorem.

Now let (a) hold. If the formation $\mathfrak{M}_1$ is l_{σ}^{τ}-irreducible, then by assumption the formations $\mathfrak{M}_1$ and $\mathfrak{L}_1$ must be contained in $\mathfrak{M}$. Therefore, $\mathfrak{X}\cap\mathfrak{M}=\mathfrak{X}_1$ and the formation $\mathfrak{F}$ satisfies condition (1) of the theorem.

If the formation l_{σ}^{τ}-reducible, then by induction the theorem holds for it. Repeating the above arguments for $\mathfrak{M}_1$ and so on, after a finite number of steps (since the $\mathfrak{H}_{\sigma}^{\tau}$-defect of the formations under consideration is finite and strictly decreasing), we obtain that $\mathfrak{X}\cap\mathfrak{M}=\mathfrak{X}_1$. Therefore, the formation $\mathfrak{F}$ satisfies condition (1) of the theorem.

Now suppose that every irreducible τ-closed σ-local subformation of $\mathfrak{F}$ with $\mathfrak{H}$-defect less than k is contained in $\mathfrak{M}$. Since $\mathfrak{F}$ is a reducible τ-closed σ-local formation, it follows that $\mathfrak{F}\setminus\mathfrak{M}$ contains a group G such that $\mathfrak{L}=l_{\sigma}^{\tau}\mathrm{form}(G)\neq\mathfrak{F}$. Then $\mathfrak{F}=\mathfrak{M}\vee_{\sigma}^{\tau}\mathfrak{L}$. By Lemma 3.2 we have $d=|\mathfrak{L}:\mathfrak{H}\cap\mathfrak{L}|_{\sigma}^{\tau}\leqslant k$. Assume that $d<k$.

If $\mathfrak{L}$ is l_{σ}^{τ}-irreducible, then by assumption $\mathfrak{L}\subseteq\mathfrak{M}$, which is impossible. This means that $\mathfrak{L}$ is a reducible τ-closed σ-local formation. But then, by induction, the theorem holds for the formation $\mathfrak{L}$. Given the assumption of irreducible τ-closed σ-local subformations with $\mathfrak{H}_{\sigma}^{\tau}$-defect less than k, and the fact that $\mathfrak{F}\cap\mathfrak{H}=\mathfrak{M}\cap\mathfrak{H}$, we again conclude that $\mathfrak{L}\subseteq\mathfrak{M}$. A contradiction. Therefore, $d=k$.

Let $\mathfrak{D}$ be an irreducible τ-closed σ-local subformation of $\mathfrak{M}$ such that $\mathfrak{D}\not\subseteq\mathfrak{L}$. By Lemma 3.2, we have $m=|\mathfrak{D}:\mathfrak{H}\cap\mathfrak{D}|_{\sigma}^{\tau}\leqslant k$. Since the formations $\mathfrak{L}$ and $\mathfrak{D}$ are contained in $\mathfrak{F}$, we have $\mathfrak{K}=\mathfrak{L}\vee_{\sigma}^{\tau}\mathfrak{D}\subseteq\mathfrak{F}$ and by Lemma 3.2 we have $d=|\mathfrak{K}:\mathfrak{H}\cap\mathfrak{K}|_{\sigma}^{\tau}\leqslant k$.

On the other hand, by Lemma 3.5 we have the equality

$$d=k+m-b,\ \text{where } b=|\mathfrak{L}\cap\mathfrak{D}:\mathfrak{H}\cap(\mathfrak{L}\cap\mathfrak{D})|_{\sigma}^{\tau}.$$

Since $\mathfrak{D}\not\subseteq\mathfrak{L}$, then $b\leqslant m-1$. Therefore, $a\geqslant k+m-(m-1)=k+1$. Contradiction. Thus, any irreducible τ-closed σ-local subformation of $\mathfrak{M}$ is contained in $\mathfrak{L}$. Therefore, if $\mathfrak{M}$ is an irreducible τ-closed σ-local formation, then $\mathfrak{M} \subseteq \mathfrak{L}$. But then $\mathfrak{F} = \mathfrak{L} \vee_\sigma^\tau \mathfrak{M} = \mathfrak{L}$, which contradicts the definition of the formation $\mathfrak{L}$. Therefore, the formation $\mathfrak{M}$ is l_σ^τ-reducible.

Suppose that $\mathfrak{L} \cap \mathfrak{H} = \mathfrak{F} \cap \mathfrak{H}$. Since $|\mathfrak{M} : \mathfrak{H} \cap \mathfrak{M}|_\sigma^\tau = k-1$, then by induction the theorem is true for the formation $\mathfrak{M}$. Therefore, the formation $\mathfrak{M}$ can be represented as (a) or (b). Given that every irreducible τ-closed σ-local non- $\mathfrak{H}$-subformation of $\mathfrak{M}$ is contained in $\mathfrak{L}$, we obtain that $\mathfrak{M} \subseteq \mathfrak{L}$. A contradiction. Thus, $\mathfrak{L} \cap \mathfrak{H} \subset \mathfrak{F} \cap \mathfrak{H}$. Since $\mathfrak{M} \cap \mathfrak{H} = \mathfrak{F} \cap \mathfrak{H}$, it follows that $\mathfrak{M} \cap \mathfrak{H} \nsubseteq \mathfrak{L} \cap \mathfrak{H}$.

Let $\mathfrak{L}$ be an irreducible τ-closed σ-local formation. Then, using the representation of the formation $\mathfrak{M}$ in the form (a) or (b) and taking into account that any irreducible τ-closed σ-local formation with $\mathfrak{H}_\sigma^\tau$-defect less than k is contained in $\mathfrak{L}$ we obtain that $\mathfrak{F} = \mathfrak{L} \vee_\sigma^\tau (\mathfrak{M} \cap \mathfrak{H})$. Thus, the formation $\mathfrak{F}$ satisfies condition (2) of the theorem.

Now let $\mathfrak{L}$ be a reducible τ-closed σ-local formation. Then, since $\mathfrak{L} \nsubseteq \mathfrak{M}$, by Theorem 4.8, $\mathfrak{L}$ contains at least one $\mathfrak{M}_\sigma^\tau$-critical formation $\mathfrak{X}$. Since any irreducible τ-closed σ-local formation with $\mathfrak{H}_\sigma^\tau$-defect less than k is contained in $\mathfrak{M}$ and $\mathfrak{M} \cap \mathfrak{H} = \mathfrak{F} \cap \mathfrak{H}$, it follows that $|\mathfrak{X} : \mathfrak{H} \cap \mathfrak{X}|_\sigma^\tau = k$. Note also that any irreducible τ-closed σ-local formation in $\mathfrak{L}$ with $\mathfrak{H}_\sigma^\tau$-defect less than k is contained in $\mathfrak{X}$, since otherwise the formation $\mathfrak{F}$ would contain an l_σ^τ-subformation with $\mathfrak{H}_\sigma^\tau$-defect greater than k, which is impossible in view of Lemma 3.1. Since the formation $\mathfrak{M}$ is maximal, we have $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{X}$. Since $\mathfrak{M} \cap \mathfrak{H} \nsubseteq \mathfrak{L} \cap \mathfrak{H}$, then $\mathfrak{M} \cap \mathfrak{H} \nsubseteq \mathfrak{X} \cap \mathfrak{H}$. Therefore, given the representation of the formation $\mathfrak{M}$ in form (a) or (b), we have $\mathfrak{F} = \mathfrak{M} \vee_\sigma^\tau \mathfrak{X} = \mathfrak{X} \vee_\sigma^\tau (\mathfrak{M} \cap \mathfrak{H})$.

Thus, the formation $\mathfrak{F}$ satisfies condition (2) of the theorem. □

In the case where $\mathfrak{H} = \mathfrak{N}_\sigma$, we obtain the following special case of Theorem 5.1.

Corollary 5.2. *Let $\mathfrak{F}$ be a reducible τ-closed σ-local formation. If and only if the σ-nilpotent l_σ^τ-defect of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F}$ satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L} \vee_\sigma^\tau \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation of σ-nilpotent l_σ^τ-defect t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is such τ-closedσ-local formation of σ-nilpotent l_σ^τ-defect $k-1$, such that $\mathfrak{L} \cap \mathfrak{M}$ is the maximal τ-closed σ-local subformation of $\mathfrak{L}$;*

(2) $\mathfrak{F} = \mathfrak{L} \vee_\sigma^\tau \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation of σ-nilpotent l_σ^τ-defect k, $\mathfrak{M}$ is a τ-closed σ-local formation such that $\mathfrak{M} \subseteq \mathfrak{H}$ and $\mathfrak{M} \nsubseteq \mathfrak{L}$.*

In particular, if $\sigma = \sigma^1 = \{\{2\}, \{3\}, \{5\}, \ldots\}$ from Theorem 5.1 we obtain

Corollary 5.3. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed local formations such that $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}$ and let $\mathfrak{F}$ be reducible. If and only if the $\mathfrak{H}_l$-defect of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F}$ satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L} \vee_l \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed local formation of $\mathfrak{H}_l$-defect t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a τ-closed local formation $\mathfrak{H}_l$-defect $k-1$ such that $\mathfrak{L} \cap \mathfrak{M}$ is the maximal τ-closed local subformation of $\mathfrak{L}$;*

(2) $\mathfrak{F} = \mathfrak{L} \vee_l \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed local formation of $\mathfrak{H}_l$-defect k, $\mathfrak{M}$ is a τ-closed local formation such that $\mathfrak{M} \subseteq \mathfrak{H}$ and $\mathfrak{M} \nsubseteq \mathfrak{L}$.*

Moreover, if $\mathfrak{H} = \mathfrak{N}$ is the formation of all nilpotent groups from Theorem 5.1, we obtain the following well-known result.

Corollary 5.4 [27]. *Let $\mathfrak{F}$ be a reducible τ-closed local formation. If and only if the nilpotent l^τ-defect of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F}$ satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L} \vee_l \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed local formation of nilpotent l^τ-defect t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a τ-closed local formation of nilpotent l^τ-defect $k-1$ such that $\mathfrak{L} \cap \mathfrak{M}$ is a maximal τ-closed local subformation of $\mathfrak{L}$;*

(2) $\mathfrak{F} = \mathfrak{L} \vee_l \mathfrak{M}$, *where $\mathfrak{L}$ is an irreducible τ-closed local formation of nilpotent l^τ-defect k, $\mathfrak{M}$ is a τ-closed local formation such that $\mathfrak{M} \subseteq \mathfrak{H}$ and $\mathfrak{M} \nsubseteq \mathfrak{L}$.*

Let $\mathfrak{F}$ – τ-closed σ-local formation. Following [2, p. 212], l_σ^τ*-length* of $\mathfrak{F}$ we define the number $l_\sigma^\tau(\mathfrak{F}) = |\mathfrak{F} : (1)|_\sigma^\tau$.

In the case when $\mathfrak{H} = (1)$, from Theorem 5.1 we obtain the following result.

Theorem 5.5. *Let $\mathfrak{F}$ be a reducible τ-closed σ-local formation. If and only if the l_σ^τ-length of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F} = \mathfrak{L} \vee_\sigma^\tau \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation l_σ^τ-length t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a τ-closed σ-local formation l_σ^τ-length $k-1$, such that $\mathfrak{L} \cap \mathfrak{M}$ is maximal τ-closed σ-local subformation of $\mathfrak{L}$.*

If τ is a trivial subgroup functor, then from Theorem 5.5 we obtain

Corollary 5.6. *Let $\mathfrak{F}$ be a reducible σ-local formation. If and only if the l_σ-length of a formation $\mathfrak{F}$ is k when $\mathfrak{F} = \mathfrak{L} \vee_\sigma \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible σ-local formation of l_σ-length t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a σ-local formation of l_σ-length $k-1$, such that $\mathfrak{L} \cap \mathfrak{M}$ is a maximal σ-local subformation of $\mathfrak{L}$.*

Let $\tau = s$ is the identity subgroup functor. Then Theorem 5.5 implies

Corollary 5.7. *Let $\mathfrak{F}$ be a reducible hereditary σ-local formation. If and only if the l^s_σ-length of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F} = \mathfrak{L} \vee^s_\sigma \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible hereditary σ-local formation of l^s_σ-length t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a hereditary σ-local formation of l^s_σ-length $k-1$, such that $\mathfrak{L} \cap \mathfrak{M}$ is a maximal hereditary σ-local subformation of the formation $\mathfrak{L}$.*

In the case where $\tau = s_n$ from Theorem 5.5 we have

Corollary 5.8. *Let $\mathfrak{F}$ be a reducible normally hereditary σ-local formation. If and only if the $l^{s_n}_\sigma$-length of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F} = \mathfrak{L} \vee^{s_n}_\sigma \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible normally hereditary σ-local formation $l^{s_n}_\sigma$-length t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a normally hereditary σ-local formation $l^{s_n}_\sigma$-length $k-1$, such that $\mathfrak{L} \cap \mathfrak{M}$ is a maximal normally hereditary formation σ-local subformation of $\mathfrak{L}$.*

In particular, if $\sigma = \sigma^1 = \{\{2\},\{3\},\{5\},\ldots\}$ from Theorem 5.5 we have

Corollary 5.9. *Let $\mathfrak{F}$ be a reducible τ-closed local formation. If and only if the l^τ-length of a formation $\mathfrak{F}$ is equal to k when $\mathfrak{F} = \mathfrak{L} \vee_l \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible τ-closed local formation of l^τ-length t, $1 \leqslant t \leqslant k-1$, and $\mathfrak{M}$ is a τ-closed local formation of l^τ-length $k-1$ such that $\mathfrak{L} \cap \mathfrak{M}$ is a maximal τ-closed local subformation of $\mathfrak{L}$.*

6. Reducible τ-closed σ-local formations of $\mathfrak{H}^\tau_\sigma$-defect 2

In this section, using Theorem 5.1, we give a description of reducible τ-closed σ-local formations with $\mathfrak{H}^\tau_\sigma$-defect 2, and also consider some special cases and consequences of the following main result of this section.

Theorem 6.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed σ-local formations such that $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$ and let $\mathfrak{F}$ be l^τ_σ-reducible. If and only if $\mathfrak{H}^\tau_\sigma$-defect of a formation $\mathfrak{F}$ is 2 when $\mathfrak{F}$ satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^\tau_\sigma \mathfrak{L}_2 \vee^\tau_\sigma \mathfrak{M}$, where $\mathfrak{M} \subseteq \mathfrak{H}$, and $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are distinct minimal τ-closed σ-local non-$\mathfrak{H}$-formations;

(2) $\mathfrak{F} = \mathfrak{L} \vee^\tau_\sigma \mathfrak{M}$, where $\mathfrak{M} \subseteq \mathfrak{H}$, and $\mathfrak{L}$ is an irreducible τ-closed σ-local formation $\mathfrak{H}^\tau_\sigma$-defect of 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

Proof. By Theorem 5.1, one of the following conditions holds for $\mathfrak{F}$:

(1) $\mathfrak{F} = \mathfrak{L} \vee^\tau_\sigma \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation $\mathfrak{H}^\tau_\sigma$-defect 1, and $\mathfrak{M}$ is a τ-closed σ-local formation $\mathfrak{H}^\tau_\sigma$-defect 1 such that $\mathfrak{L} \cap \mathfrak{M}$ is a maximal τ-closed σ-local subformation of $\mathfrak{L}$;

(2) $\mathfrak{F} = \mathfrak{L} \vee^\tau_\sigma \mathfrak{M}$, where $\mathfrak{L}$ is an irreducible τ-closed σ-local formation $\mathfrak{H}^\tau_\sigma$-defect 2, $\mathfrak{M}$ is a τ-closed σ-local formation such that $\mathfrak{M} \subseteq \mathfrak{H}$ and $\mathfrak{M} \nsubseteq \mathfrak{L}$.

Let $\mathfrak{F}$ be a formation satisfying condition (1). Since $\mathfrak{L}$ is an irreducible τ-closed σ-local formation of $\mathfrak{H}^\tau_\sigma$-defect 1, $\mathfrak{L}$ is the minimal τ-closed σ-local non- $\mathfrak{H}$-formation. Moreover, since $\mathfrak{L} \cap \mathfrak{M}$ is the maximal τ-closed σ-local subformation of $\mathfrak{L}$, it follows that $\mathfrak{L} \cap \mathfrak{M} \subseteq \mathfrak{H}$. By Theorem 4.12 we have $\mathfrak{M} = \mathfrak{M}_1 \vee^\tau_\sigma \mathfrak{L}_1$, where $\mathfrak{M}_1$ is a τ-closed σ-local subformation of $\mathfrak{H}$, $\mathfrak{L}_1$ is the minimal τ-closed σ-local non- $\mathfrak{H}$-formation. Note also that since $\mathfrak{L} \nsubseteq \mathfrak{M}$, then $\mathfrak{L} \neq \mathfrak{L}_1$. Means,

$$\mathfrak{F} = \mathfrak{L} \vee^\tau_\sigma \mathfrak{M} = \mathfrak{L} \vee^\tau_\sigma (\mathfrak{M}_1 \vee^\tau_\sigma \mathfrak{L}_1) = \mathfrak{L} \vee^\tau_\sigma \mathfrak{L}_1 \vee^\tau_\sigma \mathfrak{M}_1,$$

where $\mathfrak{M}_1 \subseteq \mathfrak{H}$, a $\mathfrak{L}$ and $\mathfrak{L}_1$ are distinct minimal τ-closed σ-local non- $\mathfrak{H}$-formations. Thus, the formation $\mathfrak{F}$ satisfies condition (1) of the theorem.

If condition (2) holds for $\mathfrak{F}$, then $\mathfrak{F}$ obviously satisfies condition (2) of the theorem. □

Theorem 6.1 has many different special cases and consequences for specific subgroup functors τ, formations $\mathfrak{H}$, and partitions σ. Let us consider some of them.

Thus, if $\tau = s$ is the identity subgroup functor, then the following holds.

Corollary 6.2. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be hereditary σ-local formations such that $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$ and let $\mathfrak{F}$ be an l^s_σ-reducible formation. If and only if $\mathfrak{H}^s_\sigma$-defect of a formation $\mathfrak{F}$ is 2 when $\mathfrak{F}$ satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^s_\sigma \mathfrak{L}_2 \vee^s_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{H}$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal s-closed* σ*-local non-*$\mathfrak{H}$*-formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^\tau_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{H}$, *and* $\mathfrak{L}$ *is an irreducible* τ*-closed* σ*-local formation* $\mathfrak{H}^s_\sigma$*-defect of* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

If $\tau(G) = s_n(G)$ is the set of all normal subgroups of G for any group G, then we obtain the following statement.

Corollary 6.3. *Let* $\mathfrak{F}$ *and* $\mathfrak{H}$ *be normally hereditary* σ*-local formations such that* $\mathfrak{F} \nsubseteq \mathfrak{H} \subseteq \mathfrak{N}_\sigma$ *and let* $\mathfrak{F}$ *be* $l^{s_n}_\sigma$*-reducible. If and only if* $\mathfrak{H}^{s_n}_\sigma$*-defect of a formation* $\mathfrak{F}$ *is* 2 *when* $\mathfrak{F}$ *satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^{s_n}_\sigma \mathfrak{L}_2 \vee^{s_n}_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{H}$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal* s_n*-closed* σ*-local non-*$\mathfrak{H}$*-formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^{s_n}_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{H}$, *and* $\mathfrak{L}$ *is an irreducible* s_n*-closed* σ*-local formation* $\mathfrak{H}^{s_n}_\sigma$*-defect of* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

In the case where $\mathfrak{H} = \mathfrak{N}_\sigma$, we obtain the following special case of Theorem 6.1.

Theorem 6.4. *Let* $\mathfrak{F}$ *be an* l^τ_σ*-reducible* τ*-closed* σ*-local formation. If and only if the* σ*-nilpotent* l^τ_σ*-defect of a formation* $\mathfrak{F}$ *is* 2 *when* $\mathfrak{F}$ *satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^\tau_\sigma \mathfrak{L}_2 \vee^\tau_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}_\sigma$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal* τ*-closed* σ*-local non-*σ*-nilpotent formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^\tau_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}_\sigma$, *and* $\mathfrak{L}$ *is an irreducible* τ*-closed* σ*-local formation with* σ*-nilpotent* l^τ_σ*-defect equal to* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

In the case where $\tau = s$ is the identity subgroup functor, Theorem 6.4 implies

Corollary 6.5. *Let* $\mathfrak{F}$ *be an* l^s_σ*-reducible hereditary* σ*-local formation. If and only if the* σ*-nilpotent* l^s_σ*-defect of a formation* $\mathfrak{F}$ *is* 2 *when* $\mathfrak{F}$ *satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^s_\sigma \mathfrak{L}_2 \vee^s_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}_\sigma$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal s-closed* σ*-local non-*σ*-nilpotent formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^s_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}_\sigma$, *and* $\mathfrak{L}$ *is an irreducible s-closed* σ*-local formation* $\mathfrak{H}^s_\sigma$*-defect of* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

If $\tau(G) = s_n(G)$, then from Theorem 6.4 we obtain

Corollary 6.6. *Let* $\mathfrak{F}$ *be an* $l^{s_n}_\sigma$*-reducible, non-*σ*-nilpotent normally hereditary* σ*-local formation. If and only if the* σ*-nilpotent* $l^{s_n}_\sigma$*-defect of a formation* $\mathfrak{F}$ *is* 2 *when* $\mathfrak{F}$ *satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^{s_n}_\sigma \mathfrak{L}_2 \vee^{s_n}_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}_\sigma$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal* s_n*-closed* σ*-local non-*σ*-nilpotent formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^{s_n}_\sigma \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}_\sigma$, *and* $\mathfrak{L}$ *is an irreducible* τ*-closed* σ*-local formation* $\mathfrak{H}^{s_n}_\sigma$*-defect of* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

In the classical case, when $\sigma = \sigma^1$, from Theorem 6.4 we obtain

Corollary 6.7 [2, Theorem 5.2.19]. *Let* $\mathfrak{F}$ *be a reducible* τ*-closed local formation. Then the nilpotent* l^τ*-defect of a formation* $\mathfrak{F}$ *is equal to* 2 *if and only if one of the following conditions holds:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee^\tau_l \mathfrak{L}_2 \vee^\tau_l \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal* τ*-closed local non-nilpotent formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^\tau_l \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}$, *and* $\mathfrak{L}$ *is a* τ^l*-irreducible* τ*-closed local formation with nilpotent* l^τ*-defect equal to* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

If, in addition, τ is a trivial subgroup functor, then we have

Corollary 6.8 [1, Theorem 20.6]. *Let* $\mathfrak{F}$ *be a reducible local formation. Then the nilpotent defect of a formation* $\mathfrak{F}$ *is equal to* 2 *if and only if* $\mathfrak{F}$ *satisfies one of the following conditions:*

(1) $\mathfrak{F} = \mathfrak{L}_1 \vee_l \mathfrak{L}_2 \vee_l \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}$, *and* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are distinct minimal local non-nilpotent formations;*

(2) $\mathfrak{F} = \mathfrak{L} \vee^\tau_l \mathfrak{M}$, *where* $\mathfrak{M} \subseteq \mathfrak{N}$, *and* $\mathfrak{L}$ *is an irreducible local formation with nilpotent defect* 2, $\mathfrak{M} \nsubseteq \mathfrak{L}$.

7. τ-Closed σ-local formations of l^{τ}_{σ}-length $\leqslant 3$

Let $\mathfrak{F}$ be a τ-closed σ-local formation. Following [2, p. 212], an *l^{τ}_{σ}-length* of a formation $\mathfrak{F}$ we define the number $l^{\tau}_{\sigma}(\mathfrak{F}) = |\mathfrak{F} : (1)|^{\tau}_{\sigma}$.

In this section, we apply Theorem 4.2 to describe τ-closed σ-local formations with l^{τ}_{σ}-length $\leqslant 3$.

Lemma 7.1. *Let $\mathfrak{F} = \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_k$, where $\mathfrak{F}_j$ is a non-identity τ-closed σ-local formation with $l^{\tau}_{\sigma}(\mathfrak{F}_j) = m_j < \infty$. Then $l^{\tau}_{\sigma}(\mathfrak{F}) = m_1 + m_2 + \cdots + m_k$. In particular, if $\mathfrak{F} \subseteq \mathfrak{N}_{\sigma}$ and $|\sigma(\mathfrak{F})| < \infty$, then $l^{\tau}_{\sigma}(\mathfrak{F}) = |\sigma(\mathfrak{F})|$.*

Proof. We prove the lemma by induction on k. For $k = 1$, the lemma is true. Now let $k \geqslant 1$ and assume that the lemma is true for $k-1$. Then, by induction, for the formation $\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1}$ we have $l^{\tau}_{\sigma}(\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1}) = m_1 + m_2 + \cdots + m_{k-1}$. By Lemma 2.13, the lattice isomorphism

$$\mathfrak{F}/\mathfrak{F}_k = \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_k /^{\tau}_{\sigma} \mathfrak{F}_k = ((\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1}) \vee^{\tau}_{\sigma} \mathfrak{F}_k) /^{\tau}_{\sigma} \mathfrak{F}_k \simeq$$

$$\simeq \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1} /^{\tau}_{\sigma} (\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1} \cap \mathfrak{F}_k) = \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1} /^{\tau}_{\sigma} (1).$$

Therefore, $l^{\tau}_{\sigma}(\mathfrak{F}) = l^{\tau}_{\sigma}(\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{k-1}) + l^{\tau}_{\sigma}(\mathfrak{F}_k) = (m_1 + m_2 + \cdots + m_{k-1}) + m_k$.

In particular, if the formation $\mathfrak{F}$ is σ-nilpotent and $|\sigma(\mathfrak{F})| < \infty$, then $\mathfrak{F} = \mathfrak{G}_{\sigma_{i_1}} \oplus \ldots \oplus \mathfrak{G}_{\sigma_{i_t}}$, where $\{\sigma_{i_1}, \ldots, \sigma_{i_t}\} = \sigma(\mathfrak{F})$. Therefore, from the first part of the lemma, we obtain $l^{\tau}_{\sigma}(\mathfrak{F}) = |\sigma(\mathfrak{F})|$. □

Lemma 7.2. *Every τ-closed σ-local formation of l^{τ}_{σ}-length 2 is reducible.*

Proof. Let $\mathfrak{F}$ be an irreducible τ-closed σ-local formation. Assume that the l^{τ}_{σ}-length of $\mathfrak{F}$ is 2. Since $\mathfrak{F} \cap \mathfrak{N}_{\sigma} = \mathfrak{N}_{\sigma(\mathfrak{F})}$ by Lemma 3.5, it follows that $|\sigma(\mathfrak{F})| = l^{\tau}_{\sigma}(\mathfrak{N}_{\sigma(\mathfrak{F})})$ by Lemma 7.1. Clearly, $|\sigma(\mathfrak{F})| > 1$. Since the formation $\mathfrak{F}$ is l^{τ}_{σ}-irreducible, it follows that $\mathfrak{N}_{\sigma(\mathfrak{F})} \subset \mathfrak{F}$. Therefore, $\mathfrak{F}$ contains a proper τ-closed σ-local subformation of l^{τ}_{σ}-length $\geqslant 2$. This contradicts Lemma 3.2. □

Lemma 7.3. *Let $\mathfrak{F}$ be a τ-closed σ-local formation. If $\mathfrak{F}$ is an l^{τ}_{σ}-irreducible formation of l^{τ}_{σ}-length 3, then $|\sigma(\mathfrak{F})| = 2$.*

Proof. Let $\mathfrak{F}$ be an l^{τ}_{σ}-irreducible formation of l^{τ}_{σ}-length 3 and $\mathfrak{M}$ be its unique maximal τ-closed σ-local subformation. Then $l^{\tau}_{\sigma}(\mathfrak{M}) = 2$. Now applying Lemmas 7.2 and 7.1 we have $|\sigma(\mathfrak{M})| = 2$. Since the formation $\mathfrak{F}$ is l^{τ}_{σ}-irreducible, any proper τ-closed σ-local subformation of $\mathfrak{F}$ is contained in $\mathfrak{M}$. Therefore, $\sigma(\mathfrak{F}) = \sigma(\mathfrak{M})$. □

Theorem 7.4. *Let $\mathfrak{F}$ be a τ-closed σ-local formation. Then the following statements hold:*

(1) $l^{\tau}_{\sigma}(\mathfrak{F}) \leqslant 2$ if and only if $\mathfrak{F} = \mathfrak{N}_{\Pi}$, where $|\Pi| \leqslant 2$;

(2) $l^{\tau}_{\sigma}(\mathfrak{F}) = 3$ if and only if $\mathfrak{F} = \mathfrak{N}_{\Pi}$, where $|\Pi| = 3$, or $\mathfrak{F}$ is a minimal τ-closed σ-local non-σ-nilpotent formation, $|\sigma(\mathfrak{F})| = 2$.

Proof. (1) If $|\sigma(\mathfrak{F})| = 1$, then $\mathfrak{F} = \mathfrak{G}_{\sigma_i}$ for some i. Then $l^{\tau}_{\sigma}(\mathfrak{F}) = 1$ by Lemma 7.1. Suppose that $\mathfrak{F}$ is not σ-nilpotent. Then $\mathfrak{F}$ is not a σ-primary formation, and hence there exist σ_i and σ_j such that $\sigma_i, \sigma_j \in \sigma(\mathfrak{F})$ $(i \neq j)$. By Lemma 2.4, we have $\mathfrak{G}_{\sigma_i}, \mathfrak{G}_{\sigma_j} \subseteq \mathfrak{F}$. Therefore, $\mathfrak{G}_{\sigma_i} \oplus \mathfrak{G}_{\sigma_j} \subseteq \mathfrak{F}$. By Lemma 7.1 we have $l^{\tau}_{\sigma}(\mathfrak{G}_{\sigma_i} \oplus \mathfrak{G}_{\sigma_j}) = 2$. Therefore, $\mathfrak{G}_{\sigma_i} \oplus \mathfrak{G}_{\sigma_j} = \mathfrak{F}$. Contradiction. This means that (1) holds.

(2) Let $l^{\tau}_{\sigma}(\mathfrak{F}) = 3$. If $\mathfrak{F}$ is σ-nilpotent, then $\mathfrak{F} = \mathfrak{N}_{\Pi}$, where $|\Pi| = 3$, by Lemmas 3.5 and 7.1. Suppose that $\mathfrak{F}$ is not σ-nilpotent. Then, by Theorem 4.10, $\mathfrak{F}$ has a minimal τ-closed σ-local non-σ-nilpotent subformation $\mathfrak{H}$. By Theorem 4.2, we have $|\sigma(\mathfrak{H})| \geqslant 2$. Since $\mathfrak{H} \cap \mathfrak{N}_{\sigma} = \mathfrak{N}_{\sigma(\mathfrak{H})}$ by Lemma 3.5, it follows that $l^{\tau}_{\sigma}(\mathfrak{H} \cap \mathfrak{N}_{\sigma}) \geqslant 2$. Since at the same time $\mathfrak{F} \not\subseteq \mathfrak{N}_{\sigma}$ and $l^{\tau}_{\sigma}(\mathfrak{F}) = 3$, then $l^{\tau}_{\sigma}(\mathfrak{H} \cap \mathfrak{N}_{\sigma}) = 2$. Therefore, $l^{\tau}_{\sigma}(\mathfrak{H}) = 3$ and $|\sigma(\mathfrak{H})| = 2$. This means $\mathfrak{H} = \mathfrak{F}$.

Let $\mathfrak{F}$ be either a τ-closed σ-local σ-nilpotent formation with $|\sigma(\mathfrak{F})| = 3$, or a minimal τ-closed σ-local non-σ-nilpotent formation and $|\sigma(\mathfrak{F})| = 2$. Then in the first case $l^{\tau}_{\sigma}(\mathfrak{F}) = 3$ by Lemma 7.1. In the second case, the formation $\mathfrak{F}$ has a unique maximal τ-closed σ-local subformation $\mathfrak{F} \cap \mathfrak{N}_{\sigma}$. By Lemma 3.5, we have $\mathfrak{F} \cap \mathfrak{N}_{\sigma} = \mathfrak{N}_{\sigma(\mathfrak{F})}$. Therefore, $|\sigma(\mathfrak{N}_{\sigma(\mathfrak{F})})| = 2$ and, therefore, $l^{\tau}_{\sigma}(\mathfrak{N}_{\sigma(\mathfrak{F})}) = 2$ by Lemma 7.1. But then $l^{\tau}_{\sigma}(\mathfrak{F}) = 3$. □

In particular, if is a trivial subgroup functor, then we have

Corollary 7.5. *Let $\mathfrak{F}$ be a σ-local formation. Then the following statements hold:*

(1) $l_{\sigma}(\mathfrak{F}) \leqslant 2$ if and only if $\mathfrak{F} = \mathfrak{N}_{\Pi}$, where $|\Pi| \leqslant 2$;

(2) $l_{\sigma}(\mathfrak{F}) = 3$ if and only if $\mathfrak{F} = \mathfrak{N}_{\Pi}$, where $|\Pi| = 3$, or $\mathfrak{F}$ is a minimal σ-local non-σ-nilpotent formation.

Using Theorem 4.2 from Theorem 7.4, we obtain the following description of τ-closed σ-local formations of l_σ^τ-length $\leqslant 3$.

Theorem 7.6. *Let $\mathfrak{F}$ be a τ-closed σ-local formation. Then the following statements hold:*

(1) $l_\sigma^\tau(\mathfrak{F}) \leqslant 2$ *if and only if* $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$, *where G is a σ-nilpotent group with* $|\sigma(G)| \leqslant 2$*;*

(2) $l_\sigma^\tau(\mathfrak{F}) = 3$ *if and only if* $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$, *where G is either a σ-nilpotent group with* $|\sigma(G)| = 3$*, or a simple non-σ-primary group with* $\tau(G) = \{1, G\}$ *and* $|\sigma(G)| = 2$*, or* $G = P \rtimes K$*, where* $P = C_G(P)$ *is a p-group,* $p \in \sigma_i$*, and K is a simple* σ_j*-group* $(j \neq i)$ *such that* $\tau(K) = \{1, K\}$.

Proof. Let $\mathfrak{F}$ be a τ-closed σ-local formation and $l_\sigma^\tau(\mathfrak{F}) \leqslant 3$. Then, by Theorem 7.4, one of the following statements holds for $\mathfrak{F}$: a) $\mathfrak{F} = \mathfrak{N}_\Pi$, where $|\Pi| \leqslant 2$; b) $\mathfrak{F} = \mathfrak{N}_\Pi$, where $|\Pi| = 3$, or $\mathfrak{F}$ is a minimal τ-closed σ-local non-σ-nilpotent formation, $|\sigma(\mathfrak{F})| = 2$.

Suppose that $\mathfrak{F}$ is σ-nilpotent, $|\sigma(\mathfrak{F})| \leqslant 3$. And let G be a σ-nilpotent group in $\mathfrak{F}$ such that $\sigma(G) = \sigma(\mathfrak{F})$. Then $l_\sigma^\tau \mathrm{form}\, G = \mathfrak{N}_{\sigma(\mathfrak{F})} = \mathfrak{F}$ by Lemma 3.5.

Now let $\mathfrak{F}$ be a minimal τ-closed σ-local non-σ-nilpotent formation, $|\sigma(\mathfrak{F})| = 2$. Then, by Theorem 4.2, we have $\mathfrak{F} = l_\sigma^\tau \mathrm{form}\, G$ and one of the following conditions holds:

1) G is a simple, non-σ-primary, τ-minimal, non-$\mathfrak{G}_{\sigma_i}$-group for any $\sigma_i \in \sigma(G)$;

2) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is a simple σ_j-group $(j \neq i)$ such that $\tau(K) = \{1, K\}$.

Let G satisfy condition 1). Then if $\sigma_i, \sigma_j \in \sigma(\mathfrak{F})$ and $H \in \tau(G) \setminus \{G\}$, then $H \in \mathfrak{G}_{\sigma_i} \cap \mathfrak{G}_{\sigma_j} = (1)$. Consequently, $\tau(G) = \{1, G\}$ and the group G satisfies Condition (2) of the theorem.

If Condition 2) is satisfied for G, then obviously G satisfies Condition (2) of the theorem. □

In particular, if τ is the trivial subgroup functor, then the following holds.

Corollary 7.7. *Let $\mathfrak{F}$ be a σ-local formation. Then the following statements hold:*

(1) $l_\sigma(\mathfrak{F}) \leqslant 2$ *if and only if* $\mathfrak{F} = l_\sigma \mathrm{form}\, G$, *where G is a σ-nilpotent group with* $|\sigma(G)| \leqslant 2$*;*

(2) $l_\sigma(\mathfrak{F}) = 3$ *if and only if* $\mathfrak{F} = l_\sigma \mathrm{form}\, G$, *where G is either a σ-nilpotent group with* $|\sigma(G)| = 3$*, or a simple non-σ-primary group with* $|\sigma(G)| = 2$*, or* $G = P \rtimes K$*, where* $P = C_G(P)$ *is a p-group,* $p \in \sigma_i$*, and K is a simple* σ_j*-group,* $j \neq i$.

In the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \{5\}, \ldots\}$ From Theorem 7.6 follows

Corollary 7.8 [2, Lemma 5.3.11]. *Let $\mathfrak{F}$ be a τ-closed local formation. Then the following statements hold:*

1) $l^\tau(\mathfrak{F}) \leqslant 2$ *if and only if $\mathfrak{F}$ is nilpotent and* $|\pi(\mathfrak{F})| \leqslant 2$*;*

2) $l^\tau(\mathfrak{F}) = 3$ *if and only if* $\mathfrak{F} = \tau^l \mathrm{form}\, G$, *where G is either a Schmidt group or a nilpotent group with* $|\pi(G)| = 3$.

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