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NON-EXPOSED FACES OF THE CONE OF COMPLETELY POSITIVE MATRICES

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Abstract. In this paper, we consider the cone of completely positive matrices. Currently, some families of non-exposed polyhedral faces of this cone were constructed. Inspired by these results, in this paper, we continue the study of the existence and properties of non-exposed faces of the cone of completely positive matrices. We prove a criterion for a face of this cone to be non-exposed. We also provide sufficient conditions that can be easily checked numerically. We show that for any $p \geqslant 6$, there exist non-exposed non-polyhedral faces of the cone of $p \times p$ completely positive matrices. Illustrative examples are given.

НЕВЫСТУПАЮЩИЕ ФАСАДЫ КОНУСА ПОЛНОСТЬЮ ПОЛОЖИТЕЛЬНЫХ МАТРИЦ

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Ключевые слова: коническая оптимизация, полностью положительные матрицы, K-полуопределенные матрицы, фасад конуса, выступающие и невыступающие фасады конуса.

Аннотация. В данной работе мы рассматриваем конус полностью положительных матриц. К настоящему время в литературе были построены некоторые семейства невыступающих полиэдральных фасадов этого конуса. Мотивированные этими результатами, в данной работе мы продолжаем изучение свойств невыступающих фасадов конуса полностью положительных матриц. Доказаны условия, выполнение которых необходимо и достаточно для того, чтобы фасад этого конуса был невыступающим. Также получены достаточные условия, которые можно легко проверить численно. Показано, что для любого $p \geqslant 6$ существуют невыступающие неполиэдральные фасады конуса $p \times p$ полностью положительных матриц. Приведены иллюстративные примеры.

1. Introduction

The optimization of linear functions in feasible sets determined by the intersection of an affine space and a convex cone is known as linear conic optimization. Conic optimization has significant applications and is a pertinent subfield of convex optimization because the conical structure of the feasible set allows one to model many nonlinearities that arise in practice. The number of such applications is increasing, and software is developing quickly (see e.g. [1; 2], and the references therein).

Since the feasible set of a linear conic program is given by the intersection of an affine space and a convex cone, the cone under consideration plays an important role and imposes the main difficulties because the objective function and all the other constraints are linear. The relevant conic problems can be solved more efficiently the better we understand a cone's structure or local description. Facial exposedness is fundamental in understanding the boundary structure of convex cones.

The copositive cone and its dual cone, namely the completely positive cone, have many applications and are actively studied for many years (see [3–6]). However, due to their complicated structures, knowledge about the geometric aspects (especially facial structures) of the copositive and completely positive cones is very limited.

In 2015, Berman et al. in [7] described the main open problems which are currently of interest in the theory of copositive and completely positive matrices. One of the stated open problems was: Is the $p \times p$ completely positive cone facially exposed?

In papers [8; 9], the author answered this question. He gave two concrete parametric classes of non-exposed polyhedral faces of the 5×5 completely positive cone constructing on the base of the Horn matrix and a semidefinite matrix in the form $T(\theta) = a(\theta)\,a'(\theta) + b(\theta)\,b'(\theta)$ with some special vectors $a(\theta), b(\theta) \in \mathbb{R}^5$ depending on a parameter $\theta \in \mathbb{R}^5$. At the end of the paper [8], the author asked "Whether there is another type of non-exposed faces in the $p \times p$ completely positive cones, or specifically in the 5×5 completely positive cones, will be an interesting research topic".

In paper [10], for any integer $p \geqslant 5$, non-exposed polyhedral faces of the cone of completely positive $p \times p$ matrices were studied. Criterion and several more explicit (easily checked) sufficient conditions for a polyhedral sub-cone of completely positive matrices to be a non-exposed face were proved. Also for any $p \geqslant 5$, the author presented new families of non-exposed faces of the cone of completely positive $p \times p$ matrices that are different from ones considered in [8; 9] for $p = 5$.

Note that all non-exposed faces considered in examples in [8–10] are polyhedral. As far as we know, there are no examples or studies in the literature of the existence of non-exposed non-polyhedral faces of the cone of completely positive matrices.

In this paper, we explore non-exposed non-polyhedral faces of the cone of completely positive matrices.

The paper is organized as follows. In Section 2 we provide some basic definitions and properties associated with a convex cone. In Section 3, necessary and sufficient conditions for a face of completely positive cone to be non-exposed are proved. Also necessary and sufficient conditions for the face to be non-polyhedral are given. In Section 4, it is shown that for any $p \geqslant 6$ there exist non-exposed non-polyhedral faces of the cone of $p \times p$ completely positive matrices. Section 5 contains some illustrative examples. The paper ends with some conclusions.

2. Definitions and notation

Given a finite dimensional space $\mathfrak{X}$, let us first remind some general definitions.

A set $C \subset \mathfrak{X}$ is convex if for any $x, y \in C$ and any $\alpha \in [0,1]$ it holds $\alpha x + (1-\alpha)y \in C$. Given a set $\mathcal{B} \subset \mathfrak{X}$, denote by conv$\mathcal{B}$ its *convex hull*, i. e., the minimal (by inclusion) convex set, containing this set, by span$(\mathcal{B})$ its span, i. e., the smallest linear subspace that contains $\mathcal{B}$, and by cone$(\mathcal{B})$ its *conic hull*. A set $K \subset \mathfrak{X}$ is a cone if for any $x \in K$ and any $\alpha > 0$, it holds $\alpha x \in K$.

A nonempty convex subset F of a convex closed set $C \subset \mathfrak{X}$ is called a *face* of C if $\alpha x + (1-\alpha)y \in F$ with $x, y \in C$ and $\alpha \in (0,1)$ imply $x, y \in F$. We say that a face F of C is *proper* if $F \neq C$ and $F \neq 0$. A face F of a closed convex set $C \subset \mathfrak{X}$ is called *exposed* if it can be represented as the intersection of C with a supporting hyperplane, i. e. there exist $y \in \mathfrak{X}$ and $d \in \mathbb{R}$ such that for all $x \in C$ it holds $\langle y, x\rangle \geqslant d$ and $\langle y, x\rangle = d$ iff $x \in F$. Every exposed face should also be a face. Given a face F of a set C, the minimal (by inclusion) exposed face containing F will be called here the *minimal exposed face* for this face.

Given a cone $K \subset \mathfrak{X}$, its dual cone K^* is given by

$$K^* = \{x \in \mathfrak{X} : \langle x, y\rangle \geqslant 0 \; \forall y \in K\}.$$

Given an integer $p > 1$, denote by $\mathbb{R}^p_+$ the set of all p vectors with non-negative components (the non-negative orthant in $\mathbb{R}^p$); let $\mathbb{S}(p)$ and $\mathbb{S}_+(p)$ be the space of real symmetric $p \times p$ matrices and the cone of symmetric positive semidefinite $p \times p$ matrices, respectively.

For a given closed subset $K \subset \mathbb{R}^p$ denote by $\mathcal{COP}(K)$ the cone of symmetric K-semidefinite matrices [11]

$$\mathcal{COP}(K) := \{D \in \mathbb{S}(p) : t^\top Dt \geqslant 0 \;\forall t \in K\}.$$

It is evident that $\mathcal{COP}(K) = \mathcal{COP}(K_{norm})$ where $K_{norm} = \{t \in K : ||t||_1 = 1\}$, $||t||_1 = |t_1| + ... + |t_p|$ for $t = (t_k, k = 1,...,p)'$.

For $K = \mathbb{R}^p_+$, we obtain the cone of copositive matrices [3]

$$\mathcal{COP}(\mathbb{R}^p_+) := \{D \in \mathbb{S}(p) : t^\top Dt \geqslant 0 \;\forall t \in \mathbb{R}^p_+\} = \mathcal{COP}(T(p)),$$

where

$$T(p) := \{t \in \mathbb{R}^p_+ : ||t||_1 = 1\}$$

is the simplex in $\mathbb{R}^p$.

For cone $\mathcal{COP}(T(p))$, the dual cone is the cone of completely positive matrices

$$\mathcal{CP}(T(p)) := (\mathcal{COP}(T(p)))^* = \text{cone}\{tt',\; t \in T(p)\}.$$

Recall that a completely positive matrix $U \in \mathcal{CP}(T(p))$ can be written as $\sum_{i\in I} t(i)t'(i)$ with some $t(i) \in \mathbb{R}^p_+,\; i \in I,\; |I| \leqslant p(p+1)/2$.

For given closed subset $K \subset \mathbb{R}^p$ and $X \in \mathcal{COP}(K)$, denote by $Z_0(X, K_{norm})$ the set of normalized zeroes of X in K:

$$Z_0(X, K_{norm}) := \{t \in K_{norm} : t'Xt = 0\}.$$

For a vector $t = (t_k, k = 1,...p)' \in T(p)$, denote by $\text{supp}(t)$ its support: $\text{supp}(t) = \{k \in \{1,...,p\} : t_k > 0\}$. For $X \in \mathcal{COP}(T(p))$, a normalized zero $\tau \in Z_0(X, T(p))$ is called a minimal zero of X if there does not exist zero $\bar{\tau} \in Z_0(X, T(p))$ such that the inclusion $\text{supp}(\bar{\tau}) \subset \text{supp}(\tau)$ holds true strictly.

In what follows we will use the following proposition.

Proposition 2.1. *Let M, $N(m)$, $m \in M$, be given finite index sets and $\xi(j) \in \mathbb{R}^p$, $j \in N(m), m \in M$, be given vectors. Set*

$$\mathcal{T} := \bigcup_{m\in M} \mathcal{T}(m) \textit{ where } \mathcal{T}(m) := \text{conv}\{\xi(j), j \in N(m)\}$$

and consider $X \in \mathcal{COP}(\mathcal{T})$. Then the set

$$Z_0(X, \mathcal{T}) := \{t \in \mathcal{T} : t'Xt = 0\}$$

of all normalized zeros of X in $\mathcal{T}$ is empty or a union of a finite number of polytopes.

Proof. It is evident that $Z_0(X, \mathcal{T}) = \bigcup_{m\in M} Z_0(X, \mathcal{T}(m))$. For $m \in M$, denote

$$p_m = |N(m)|,\; W(m) = (\xi(j), j \in N(m)),\; B(m) = W(m)'XW(m).$$

It is easy to see that

$$X \in \mathcal{COP}(\mathcal{T}(m)) \iff B(m) \in \mathcal{COP}(T(p_m)).$$

Let $Z_0(B(m), T(p_m)) := \{t \in T(p_m) : t'B(m)t = 0\}$ be the set of normalized zeros of $B(m)$ in $T(p_m)$ and $\{\tau(m, j), j \in J(m)\}$ be the set of normalized minimal zeros of $B(m)$ in $T(p_m)$. It is known (see Proposition 2.3 and its proof with $A_* = B(m)$ in [12]) that there exists the set $\{J(s,m), s \in S(m)\}$ of subsets of $J(m)$ such that

$$Z_0(B(m), T(p_m)) = \bigcup_{s\in S(m)} \text{conv}\{\tau(m, j), j \in J(s,m)\}.$$

Then it is easy to see that

$$Z_0(X, \mathcal{T}(m)) = \bigcup_{s\in S(m)} \text{conv}\{\bar{\tau}(m, j), j \in J(s,m)\},$$

where $\bar{\tau}(m,j) = W(m)\tau(m,j)$ for all $j \in J(m)$. Consequently,

$$Z_0(X,\mathcal{T}) = \bigcup_{m\in M}\bigcup_{s\in S(m)} Z_{ms}(X,\mathcal{T}),$$

where $Z_{ms}(X,\mathcal{T}) := \text{conv}\{\bar{\tau}(m,j), j \in J(s,m)\}$ for $s \in S(m)$, $m \in M$, are polytopes. □

It follows from Proposition 2.1 that if K_{norm} is a union of a finite number m_* of polytopes, then $Z_0(X,K)$ is empty or a union of a finite number m_0 of polytopes. Note that, as a rule, $m_0 \neq m_*$.

3. Non-exposed faces

In this section, we formulate and prove a criterion for a proper face of $\mathcal{CP}(T(p))$ to be non-exposed without assumption that the face is polyhedral.

Let $\mathcal{F}$ be a proper face of $\mathcal{CP}(T(p))$. Then applying Algorithm 2 from [13], we conclude that there exists a finite integer $k_0 \geqslant 0$ such that $\mathcal{F} = \mathcal{F}_{k_0+1}$, where

$$\mathcal{F}_0 = \mathcal{CP}(T(p)),\ X_k \in \mathcal{F}_k^*,\ X_k \notin \mathcal{F}_k^\perp,\ \mathcal{F}_{k+1} = \mathcal{F}_k \cap X_k^\perp,\ k = 0,...,k_0.$$

It follows from these relations that

$$\begin{gathered} X_0 \in \mathcal{COP}(T(p)),\ Z_0(0) := Z_0(X_0,T(p)) \neq \varnothing,\ Z_0(0) \neq T(p), \\ X_{k+1} \in \mathcal{COP}(Z_0(k)),\ Z_0(k+1) := Z_0(X_{k+1},Z_0(k)) \neq \varnothing,\ Z_0(k+1) \neq Z_0(k), \\ k = 0,...,k_0-1; \end{gathered} \tag{3.1}$$

$$\mathcal{F} = \{U = \sum_{i=1}^{p(p+1)/2} \alpha_i t(i)t'(i) : \alpha_i \geqslant 0,\ t(i) \in Z_0(k_0)\ \forall i = 1,...,p(p+1)/2\}. \tag{3.2}$$

On the other hand, let $\{X_k, k = 0,...,k_0\}$ be a set of matrices and $\{Z_0(k), k = 0,...,k_0\}$ be a set of subsets of $T(p)$ satisfying (3.1), then the set $\mathcal{F}$ defined by the rule (3.2) is a proper face of $\mathcal{CP}(T(p))$. This gives us a way for constructing proper faces of $\mathcal{CP}(T(p))$.

It is evident that if $k_0 = 0$, then the face (3.2) is an exposed face of $\mathcal{CP}(T(p))$.

Corollary 3.1. *Let $\{X_k, k = 0,...,k_0\}$ be a set of matrices and $\{Z_0(k), k = 0,...,k_0\}$ be a set of subsets of $T(p)$ satisfying* (3.1)*. Then for any $k = 0,...,k_0$, the set $Z_0(k)$ is non-empty and is a union of a finite number of polytopes. Hence the set $Z_0(k_0)$ in* (3.2) *can be represented in the form*

$$Z_0(k_0) =: \mathcal{Z} = \bigcup_{s\in\mathcal{S}} \mathcal{Z}(s),\ \mathcal{Z}(s) = \text{conv}\{\xi(j), j \in \mathcal{J}(s)\} \tag{3.3}$$

with certain finite index sets $\mathcal{J}(s), s \in \mathcal{S}$, and vectors $\xi(j) \in T(p),\ j \in \mathcal{J}(s),\ s \in \mathcal{S}$.

Proof. By construction, $Z_0(k_0) \subset Z_0(k_0-1) \subset ... \subset Z_0(1) \subset Z_0(0) \subset T(p)$, and for any $k = 0,...,k_0$, the set $Z_0(k)$ is non-empty.

Applying Proposition 2.1 with $m = 1$ and $\mathcal{T} = \mathcal{T}(1) = T(p)$, we conclude that the set $Z_0(0) := := Z_0(X_0,T(p))$ is a union of a finite number of polytopes. After applying Proposition 2.1 successively with $\mathcal{T} = Z_0(k)$ for $k = 0,...,k_0$, we obtain that for any $k = 0,...,k_0$, the set $Z_0(k)$ is a union of a finite number of polytopes. Consequently, the set $Z_0(k_0)$ in (3.2) can be represented in the form (3.3) with certain finite index sets $\mathcal{J}(s), s \in \mathcal{S}$, and vectors $\xi(j) \in T(p),\ j \in \mathcal{J}(s),\ s \in \mathcal{S}$. □

In what follows, we will use the following lemma.

Lemma 3.2. *Let $t \in \mathbb{R}^p_+ \setminus \{\mathbf{0}\}$, $\mu(i) \in T(p)$, $i \in I$, $|I| < \infty$, be given and*

$$t\,t' = \sum_{i\in I} \alpha_i \mu(i)(\mu(i))' \text{ with some } \alpha_i \geqslant 0\ \forall i \in I. \tag{3.4}$$

Then there exist $\beta > 0$ and $i_* \in I$ such that $t = \beta\mu(i_*)$.

Proof. Denoting $B = (\sqrt{\alpha_i}\mu(i), i \in I)$, we can rewrite the equality from (3.4) in the form $t\,t' = BB'$ whereform we conclude that

$$1 = \text{rank}\,(t\,t') = \text{rank}\,(BB').$$

It follows from the equality $\operatorname{rank}(BB') = 1$ that $\operatorname{rank}(B) = 1$ and hence there exist an index $i_* \in I$ and numbers $\beta_i \geqslant 0$, $i \in I$, such that $\sqrt{\alpha_i}\mu(i) = \beta_i\mu(i_*)$ for all $i \in I$. Then equality in (3.4) takes the form

$$tt' = \tilde{\beta}\mu(i_*))\,\mu'(i_*),$$

where $\tilde{\beta} = \sum_{i\in I}\beta_i^2 > 0$. This implies that $t = \beta\mu(i_*)$ with $\beta = \sqrt{\tilde{\beta}} > 0$. □

Proposition 3.3. *Let $k_0 \geqslant 1$ and $\{X_k, k = 0,...,k_0\}$ be a set of matrices satisfying* (3.1)*, then the set $\mathcal{F}$ defined by the rule* (3.2) *is a non-exposed face of $\mathcal{CP}(T(p))$ iff there exists $t^* \in Z_0(X_0,T(p)) \setminus Z_0(k_0)$ such that for any $D \in \mathcal{COP}(T(p))$*

$$\textit{the equalities } t'Dt = 0\ \forall t \in Z_0(k_0) \textit{ imply the equality } t^{*\prime}Dt^* = 0. \tag{3.5}$$

Proof. Let $\mathcal{F}$ be a face of $\mathcal{CP}(T(p))$ having representation (3.2) and suppose that there exists $t^* \in Z_0(X_0,T(p)) \setminus Z_0(k_0)$ such that for any $D \in \mathcal{COP}(T(p))$, condition (3.5) holds true. Suppose the contrary: the face $\mathcal{F}$ is an exposed face of $\mathcal{CP}(T(p))$. Then there exists a matrix $A \in \mathcal{COP}(T(p))$ such that

$$\mathcal{F} = \mathcal{CP}(T(p)) \cap A^{\perp} =$$
$$= \{U = \sum_{i=1}^{p_*}\bar{\alpha}_i\mu(i)\mu(i)',\ \bar{\alpha}_i \geqslant 0,\ \mu(i) \in Z_0(A,T(p))\ \forall i = 1,...,p_*\}, p_* = p(p+1)/2. \tag{3.6}$$

Let us show that

$$Z_0(A,T(p)) = Z_0(k_0). \tag{3.7}$$

Consider any $t \in Z_0(A,T(p))$. Then it follows from (3.6) that $tt' \in \mathcal{F}$ and it follows from the representation (3.2) that

$$tt' = \sum_{i\in I_*}\alpha_i t(i)t(i)' \text{ with some } \alpha_i > 0,\ t(i) \in Z_0(k_0)\ \forall i \in I_* \subset \{1,...,p_*\}.$$

Then it follows from Lemma 3.2 that $t = \beta t(i_0)$ with some $i_0 \in I_*$ and $\beta > 0$ and taking into account that $t \in Z_0(A,T(p))$ and $t(i_0) \in Z_0(k_0)$, we obtain that $t = t(i_0) \in Z_0(k_0)$. Consequently, the inclusion $Z_0(A,T(p)) \subset Z_0(k_0)$ holds true.

Now consider any $t \in Z_0(k_0)$. This inclusion implies that $tt' \in \mathcal{F}$, and it follows from (3.6) that

$$tt' = \sum_{i\in\bar{I}_*}\bar{\alpha}_i\mu(i)\mu(i)',\ \bar{\alpha}_i > 0,\ \mu(i) \in Z_0(A,T(p))\ \forall i \in \bar{I}_* \subset \{i = 1,...,p_*\}.$$

Then it follows from Lemma 3.2 that $t = \beta\mu(i_0)$ with some $i_0 \in \bar{I}_*$ and $\beta > 0$ and taking into account that $t \in Z_0(k_0)$ and $\mu(i_0) \in Z_0(A,T(p))$, we obtain that $t = \mu(i_0) \in Z_0(A,T(p))$. Consequently, the inclusion $Z_0(k_0) \subset Z_0(A,T(p))$ holds true. Hence the equality (3.7) takes place.

It follows from (3.7) that for $A \in \mathcal{COP}(T(p))$ equalities $t'At = 0\ \forall t \in Z_0(k_0)$ hold true. Taking into account that for any $D \in \mathcal{COP}(T(p))$, condition (3.5) holds true, we obtain that $t^{*\prime}At^* = 0$, and consequently, $t^* \in Z_0(A,T(p))$. This inclusion and (3.7) imply that $t^* \in Z_0(k_0)$. But by construction, $t^* \in Z_0(X_0,T(p)) \setminus Z_0(k_0)$. Hence our assumption that $\mathcal{F}$ is an exposed face is wrong.

Now suppose that a face $\mathcal{F}$ represented in the form (3.2) is a non-exposed face of $\mathcal{CP}(T(p))$. Let $\mathcal{F}_{min} := \mathcal{CP}(T(p)) \cap X_{min}^{\perp}$, $X_{min} \in \mathcal{COP}(T(p))$, be the minimal exposed face of $\mathcal{CP}(T(p))$ containing the face $\mathcal{F}$. Since $\mathcal{F} \subset \mathcal{F}_{min}$ and $\mathcal{F} \neq \mathcal{F}_{min}$, there exists $U^* \in \mathcal{F}_{min}$ such that $U^* \notin \mathcal{F}$.

It follows form the inclusion $U^* \in \mathcal{F}_{min}$ that U^* admits a representation

$$U^* = \sum_{i\in I_*}\alpha_i^* t^*(i)t^*(i)' \text{ with some } \alpha_i^* > 0,\ t^*(i) \in Z_0(X_{min},T(p))\ \forall i \in I_*.$$

Then the condition $U^* \notin \mathcal{F}$ implies that there exists $i_0 \in I_*$ such that

$$t^*(i_0) \in Z_0(X_{min},T(p)),\ t^*(i_0) \notin Z_0(k_0).$$

Let us show that the condition (3.5) holds true with $t^* = t^*(i_0)$. Suppose the contrary: there exists $B \in \mathcal{COP}(T(p))$ such that

$$t'Bt = 0\ \forall t \in Z_0(k_0),\ t^{*\prime}Bt^* > 0. \tag{3.8}$$

Consider a matrix $\bar{X} := X_{min} + B \in \mathcal{COP}(T(p))$. It follows from (3.8) that

$$Z_0(k_0) \subset Z_0(\bar{X}, T(p)) \subset Z_0(X_{min}, T(p)),\ t^* \in Z_0(X_{min}, T(p)),\ t^* \notin Z_0(\bar{X}, T(p)).$$

But these relations contradict the assumption that $\mathcal{F}_{min} = \mathcal{CP}(T(p)) \cap X_{min}^{\perp}$, is the minimal exposed face of $\mathcal{CP}(T(p))$ containing the face $\mathcal{F}$. Thus we have shown that condition (3.5) holds true with $t^* = t^*(i_0)$.

Now let us show that $t^* = t^*(i_0) \in Z_0(X_0, T(p))$. Suppose the contrary: $t^*(i_0) \notin Z_0(X_0, T(p))$. Consider a matrix $\widetilde{X} := X_{min} + X_0 \in \mathcal{COP}(T(p))$. Then, taking into account that by construction, $\mathcal{F} \subset \subset \mathcal{CP}(T(p)) \cap X_{min}^{\perp}$ and $\mathcal{F} \subset \mathcal{CP}(T(p)) \cap X_0^{\perp}$, we obtain

$$\mathcal{F} \subset \mathcal{CP}(T(p)) \cap \widetilde{X}^{\perp} \subset \mathcal{CP}(T(p)) \cap X_{min}^{\perp},\ t^*(i_*) \in Z_0(X_{min}, T(p)),\ t^*(i_*) \notin Z_0(\widetilde{X}, T(p)).$$

But these relations contradict the assumption that $\mathcal{F}_{min} = \mathcal{CP}(T(p)) \cap X_{min}^{\perp}$, is the minimal exposed face of $\mathcal{CP}(T(p))$ containing the face $\mathcal{F}$. Thus we have shown that $t^* = t^*(i_0) \in Z_0(X_0, T(p))$.

Hence, we have proved that if $\mathcal{F}$ is a non-exposed face of $\mathcal{CP}(T(p))$, then there exists $t^* \in \in Z_0(X_0, T(p)) \setminus Z_0(k_0)$ such that for any $D \in \mathcal{COP}(T(p))$ condition (3.5) holds true. □

A cone $C \subset \mathbb{R}^n$ is said to be a polyhedral if there exists a finite set $\{c(i), i \in I\}$ of vectors $c(i) \in \mathbb{R}^n$, $||c(i)||_1 = 1$, $i \in I$, such that $C = \text{cone}\{c(i)c(i)', i \in I\}$.

Proposition 3.4. *For a given proper face $\mathcal{F}$ of $\mathcal{CP}(T(p))$, let $\mathcal{Z} := Z_0(k_0)$ be a subset of $T(p)$ such that $\mathcal{F}$ admits the representation* (3.2). *Then the face $\mathcal{F}$ is polyhedral iff the set $\mathcal{Z}$ consists of a finite number of elements, i. e., $\mathcal{Z} = \{\tau(j),\ j \in J\}$ with certain index set J, $|J| < \infty$, and vectors $\tau(j) \in T(p)$, $j \in J$.*

Proof. Suppose that the set $\mathcal{Z}$ consists of a finite number of elements, i. e., $\mathcal{Z} = \{\tau(j),\ j \in J\}$. Then it is evident that

$$\mathcal{F} = \{U = \sum_{i=1}^{p(p+1)/2} \alpha_i t(i)t(i)' : \alpha_i \geqslant 0,\ t(i) \in \mathcal{Z}\ \forall i = 1, ..., p(p+1)/2\} = \text{cone}\{\tau(j)\tau(j)', j \in J\},$$

and consequently, the face $\mathcal{F}$ is polyhedral.

Now suppose that a face $\mathcal{F}$ of $\mathcal{CP}(T(p))$ having representation (3.2) is polyhedral. Hence there exist a finite set J and vectors $\tau(j) \in T(p)$, $j \in J$, such that

$$\mathcal{F} = \text{cone}\{\tau(j)\tau(j)',\ j \in J\}. \tag{3.9}$$

Hence, for $j \in J$, we have $\tau(j)\tau(j)' \in \mathcal{F}$, and it follows from (3.2) that

$$\tau(j)\tau(j)' = \sum_{i \in I(j)} \alpha_{ij} t(i)t(i)' \text{ with some } \alpha_{ij} > 0,\ t(i) \in \mathcal{Z}\ \forall i \in I(j).$$

Then applying Lemma 3.2 we conclude that there exists $i_j \in I(j)$ and $\beta(j) > 0$ such that $\tau(j) = \beta(j)t(i_j)$. Taking into account that $\tau(j) \in T(p)$ and $t(i_j) \in T(p)$, we obtain that $\tau(j) = t(i_j)$ and consequently $\tau(j) = t(i_j) \in \mathcal{Z}$. Thus, $\{\tau(j),\ j \in J\} \subset \mathcal{Z}$.

Now consider $t \in \mathcal{Z}$. There it follows from (3.2) that $tt' \in \mathcal{F}$, and it follows from (3.9) that

$$tt' = \sum_{j \in J_*} \alpha_j^* \tau(j)\tau(j)' \text{ with some } \alpha_j > 0,\ j \in J_* \subset J.$$

Then applying Lemma 3.2 and taking into account that $t \in T(p)$ and $\tau(j) \in T(p)$ for all $j \in J$, we conclude that there exists $j_0 \in J_*$ such that $t = \tau(j_0)$. This implies that $t \in \{\tau(j), j \in J\}$ for any $t \in \mathcal{Z}$. Hence $\mathcal{Z} \subset \{\tau(j), j \in J\}$. It follows from this inclusion and the inclusion $\{\tau(j), j \in J\} \subset \mathcal{Z}$ proved above that $\mathcal{Z} = \{\tau(j), j \in J\}$. □

Corollary 3.5. *For a given proper face $\mathcal{F}$ of $\mathcal{CP}(T(p))$, let $\mathcal{Z} := Z_0(k_0)$ be a subset of $T(p)$ such that $\mathcal{F}$ admits the representation* (3.2). *Then the face $\mathcal{F}$ is non-polyhedral iff there exist $\tau \in \mathcal{Z}$ and $\mu \in \mathcal{Z}$ such that $\tau \neq \mu$, $\alpha\tau + (1-\alpha)\mu \in \mathcal{Z}$ for all $\alpha \in [0, 1]$.*

Proof. Suppose that there exist $\tau \in \mathcal{Z}$ and $\mu \in \mathcal{Z}$ such that $\tau \neq \mu$, $\alpha\tau + (1-\alpha)\mu \in \mathcal{Z}$ for all $\alpha \in \in [0, 1]$. Then, it is evident that the set $\mathcal{Z}$ consists of an infinite number of elements, and it follows from Proposition 3.4 that the face $\mathcal{F}$ is non-polyhedral.

Now suppose that the face $\mathcal{F}$ is non-polyhedral. Then it follows from Proposition 3.4 that the set $\mathcal{Z}$ consists of an infinite number of elements. Moreover, due to Proposition 2.1, we know that the set $\mathcal{Z}$ is a union of a finite number of polytopes. Taking into account these facts, one conclude that there exist $\tau \in \mathcal{Z}$ and $\mu \in \mathcal{Z}$ such that $\tau \neq \mu$, $\alpha\tau + (1-\alpha)\mu \in \mathcal{Z}$ for all $\alpha \in [0,1]$. □

3.1. Sufficient condition for a proper face to be non-exposed

In Proposition 3.3, for a given $t^* \in T$, it is not easy to test if for any $D \in \mathcal{COP}(T(p))$, condition (3.5) holds true. In this subsection, we give easily tested sufficient conditions for fulfillment of this condition.

For a given set $\{X_k, k = 0,...,k_0\}$ of matrices satisfying (3.1), consider the set $\mathcal{F}$ defined by the rule (3.2). It was shown above that the set $\mathcal{F}$ is a proper face of $\mathcal{CP}(T(p))$. It was shown above that the set $Z_0(k_0) =: \mathcal{Z}$ is a union of a finite number of polytopes.

Knowing the set of matrices $\{X_k, k = 0,...,k_0\}$ and consistently using Lemma 2.1 and the algorithms from [14], we can find the sets $\mathcal{J}(s)$, $s \in \mathcal{S}$, and vectors $\xi(j) \in T(p), j \in \mathcal{J}(s)$, $s \in \mathcal{S}$, such that the set $Z_0(k_0)$ admits the representation

$$Z_0(k_0) = \bigcup_{s\in\mathcal{S}} \mathcal{Z}(s),\ \mathcal{Z}(s) = \text{conv}\{\xi(j), j \in \mathcal{J}(s)\}. \tag{3.10}$$

Denote $\mathcal{J} := \bigcup_{s\in\mathcal{S}} \mathcal{J}(s)$, $\mathcal{S}(j) = \{s \in \mathcal{S} : j \in \mathcal{J}(s)\}$ for all $j \in \mathcal{J}$,

$$P_*(s) := \bigcup_{j\in\mathcal{J}(s)} \text{supp}(\xi(j))\ \forall s \in \mathcal{S};\ M_*(j) := \bigcup_{s\in\mathcal{S}(j)} P_*(s)\ \forall j \in \mathcal{J}.$$

Proposition 3.6. *Consider a set of matrices $\{X_k, k = 0,...,k_0\}$ and the corresponding set $\{Z_0(k), k = = 0,...,k_0\}$ of subsets of $T(p)$ satisfying (3.1). Suppose that the representation (3.10) of the set $Z_0(k_0)$ is known and let $M_*(j)$, $\xi(j)$, $j \in J$, be the corresponding sets and vectors defined above.*

Suppose that there exists $t^* \in Z_0(0) \setminus Z_0(k_0)$ such that for any $D \in \mathbb{S}(p)$, the qualities

$$e_k' D\xi(j) = 0\ \forall k \in M_*(j),\ \forall j \in \mathcal{J}, \tag{3.11}$$

imply the equalities

$$e_k' Dt^* = 0\ \forall k \in \text{supp}(t^*). \tag{3.12}$$

Then the set $\mathcal{F}$ defined in (3.2) is a non-exposed face of $\mathcal{CP}(T(p))$. Here $\{e_k, k = 1,...,p\}$ is the standard basis in $\mathbb{R}^p$.

Proof. Consider the sets of matrices

$$\mathcal{A} := \{D \in \mathcal{COP}(T(p)) : t'Dt = 0\ \forall t \in Z_0(k_0)\},$$
$$\mathcal{B} := \{D \in \mathbb{S}(p) : e_k' D\xi(j) = 0\ \forall k \in M_*(j),\ \forall j \in J\}.$$

Let us show that

$$\mathcal{A} \subset \mathcal{B}. \tag{3.13}$$

It is evident that it follows from the inclusion $D \in \mathcal{A}$ and representation (3.10) of the set $Z_0(k_0)$ that

$$\xi(j)' D\xi(j) = 0\ \forall j \in \mathcal{J}. \tag{3.14}$$

It is easy to see that these equalities and condition $D \in \mathcal{COP}(T(p))$ imply the inequalities

$$D\xi(j) \geqslant\ \forall j \in \mathcal{J}.$$

Let us show that

$$\xi(i)' D\xi(j) = 0\ \forall i \in \mathcal{J}(s), \forall j \in \mathcal{J}(s),\ \forall s \in \mathcal{S},\ \forall D \in \mathcal{A}. \tag{3.15}$$

In fact, it follows from the inclusion $D \in \mathcal{A}$ and (3.10) that

$$t'Dt = 0\ \forall t \in \mathcal{Z}(s) = \text{conv}\{\xi(j), j \in \mathcal{J}(s)\},\ \forall s \in \mathcal{S}.$$

Hence, taking into account (3.14) and inclusions $0.5(\xi(i) + \xi(j)) \in \mathcal{Z}(s)$ for all $i \in \mathcal{J}(s)$, $j \in \mathcal{J}(s)$ and $s \in \mathcal{S}$, we obtain $0 = (\xi(i) + \xi(j))' D(\xi(i) + \xi(j)) = 2\xi(i)' D\xi(j)$ for all $i \in \mathcal{J}(s)$, $j \in \mathcal{J}(s)$ and $s \in \mathcal{S}$. Thus, we have shown that equalities (3.15) hold true.

Now we will show that

$$e_k' D\xi(j) = 0 \ \forall k \in P_*(s),\ \forall j \in \mathcal{J}(s),\ \forall s \in \mathcal{S},\ \forall D \in \mathcal{A}. \tag{3.16}$$

Suppose contrary: there exist $D \in \mathcal{A}$, $s_0 \in \mathcal{S}$, $k_0 \in P_*(s_0)$ and $j_0 \in \mathcal{J}(s_0)$ such that

$$e_{k_0}' D\xi(j_0) > 0.$$

Since $k_0 \in P_*(s_0)$, there exists an index $i_0 \in \mathcal{J}(s_0)$ such that $k_0 \in \text{supp}(\xi(i_0))$. Taking into account the inequalities $\xi(i_0) \geqslant \mathbf{0}$, $D\xi(j_0) \geqslant \mathbf{0}$ and $\xi_{k_0}(i_0) > 0$, let us calculate

$$\xi(i_0)' D\xi(j_0) = \sum_{k \in P} \xi_k(i_0) e_k' D\xi(j_0) \geqslant \xi_{k_0}(i_0) e_{k_0}' D\xi(j_0) > 0.$$

But this contradicts equalities (3.15). Thus, we have shown that that equalities (3.16) hold true.

Now let us consider any $j \in \mathcal{J}$ and the corresponding set $\mathcal{S}(j)$. It follows from (3.16) and the definition of the set $\mathcal{S}(j)$ that for any $D \in \mathcal{A}$ we have

$$e_k' D\xi(j) = 0 \ \forall k \in P_*(s),\ \forall s \in \mathcal{S}(j) \iff e_k' D\xi(j) = 0 \ \forall k \in \bigcup_{s \in \mathcal{S}(j)} P_*(s).$$

Taking into account the latter equalities and the definition of the set $M_*(j)$, we conclude that $e_k' D\xi(j) = 0 \ \forall k \in M_*(j),\ \forall j \in \mathcal{J},\ \forall D \in \mathcal{A}$. This implies inclusion (3.13).

It follows from inclusion (3.13) that the conditions of this proposition guarantee the fulfillment of conditions of Proposition 3.3. Wherefrom we conclude that the face $\mathcal{F}$ under consideration is a non-exposed face of $\mathcal{CP}(T(p))$. □

In contrast to conditions of Proposition 3.3, the conditions of Proposition 3.6 can be easily verified.

In fact, for any matrix $D \in \mathbb{S}(p)$ with elements $d_{ij}, i = 1,...,p,\ j = i,...,p$, we define the vector $\text{svec}(D) \in \mathbb{R}^{p(p+1)/2}$ by the rule (see [15])

$$\text{svec}(D) = (d_{11}, \sqrt{2}d_{12}, ..., \sqrt{2}d_{1p}, d_{22}, \sqrt{2}d_{23}, ..., \sqrt{2}d_{2p}, ..., d_{pp})^{\top}.$$

Suppose that $\xi(j), M_*(j),\ j \in J$, and $t^* \in Z_0(0) \setminus Z_0(k_0)$ are given. Then it is easy to construct matrices $\mathbb{B} \in \mathbb{R}^{p_a \times p(p+1)/2}$ with $p_a = \sum_{j \in J} |M_*(j)|$ and $\mathbb{W} \in \mathbb{R}^{p_b \times p(p+1)/2}$ with $p_b = |\text{supp}(t^*)|$ such that

$$\mathcal{B} = \{D \in \mathbb{S}(p) : \mathbb{B}\,\text{svec}(D) = \mathbf{0}\},$$

$$\{D \in \mathbb{S}(p) : e_k' D t^* = 0 \ \forall k \in \text{supp}(t^*)\} = \{D \in \mathbb{S}(p) : \mathbb{W}\,\text{svec}(D) = \mathbf{0}\}.$$

Then it is evident that for any $D \in \mathbb{S}(p)$, the qualities (3.11) imply the equalities (3.12) if and only if

$$\text{rank}\,\mathbb{B} = \text{rank}\begin{pmatrix} \mathbb{B} \\ \mathbb{W} \end{pmatrix}. \tag{3.17}$$

The latter condition can be easily tested by any available algorithm calculating the rank of a given matrix.

4. Non-exposed non-polyhedral faces generated by a non-exposed polyhedral face

Theorem 4.1. *Let* $\mathcal{F} := \text{cone}\{\tau(s)\tau(s)', s \in \mathcal{S}\}$ *with* $|\mathcal{S}| < \infty$, $\tau(s) \in T(p)$ *for all* $s \in \mathcal{S}$, *be a non-exposed polyhedral face of* $\mathcal{CP}(T(p))$. *Then for any integer* $q > 0$, *the set*

$$\mathcal{F}_* := \text{cone}\{\bar{t}\bar{t}', \bar{t} \in \bigcup_{s \in \mathcal{S}} T_*(s)\},$$

where $T_*(s) = \text{conv}\{\bar{\tau}(s), \mathbf{e}_{p+k}, k = 1,...,q\}$, $\bar{\tau}(s) = (\tau'(s), 0, ..., 0)' \in \mathbb{R}^{p+q}\ \forall s \in \mathcal{S}$, $\{\mathbf{e}_k, k = 1,...,p+q\}$ *is the standard basis in* $\mathbb{R}^{p+q}$, *is a non-exposed non-polyhedral face of* $\mathcal{CP}(T(p+q))$ *with* $T(p+q) = \{t \in \mathbb{R}^{p+q}_+ : ||t||_1 = 1\}$.

Proof. Since $\mathcal{F}$ is a face of $\mathcal{CP}(T(p))$, the following holds true

$$A \in \mathcal{CP}(T(p)),\ B \in \mathcal{CP}(T(p)),\ A + B \in \mathcal{F} \implies A \in \mathcal{F},\ B \in \mathcal{F}. \tag{4.1}$$

By assumption $\mathcal{F}$ is a non-exposed polyhedral face of $\mathcal{CP}(T(p))$, hence, it follows from [10] that there exists $\tau_* \in T(p) \setminus \{\tau(s),\, s \in \mathcal{S}\}$ such that, for any $D \in \mathcal{COP}(T(p))$, the equalities

$$\tau'(s)D\tau(s) = 0 \;\forall s \in \mathcal{S}, \tag{4.2}$$

imply the equality

$$\tau_*{}'D\tau_* = 0. \tag{4.3}$$

Let us show that $\mathcal{F}_*$ is a face of $\mathcal{CP}(T(p+q))$.

Notice that, for any $s \in \mathcal{S}$,

$$\bar{t} \in T_*(s) \iff \bar{t} = \begin{pmatrix} \alpha\tau(s) \\ t \end{pmatrix} \text{ with some } \alpha \geqslant 0,\; t \in \mathbb{R}^q_+,\; \alpha + ||t||_1 = 1. \tag{4.4}$$

Consequently, if $U_* \in \mathcal{F}_*$ then U_* admits a presentation

$$U_* = \sum_{s\in\mathcal{S}} \begin{pmatrix} \alpha_s\tau(s) \\ t(s) \end{pmatrix} (\alpha_s\tau'(s), t'(s)) \text{ with some } \alpha_s \geqslant 0,\; t(s) \in \mathbb{R}^q_+ \;\forall s \in \mathcal{S}. \tag{4.5}$$

Suppose that

$$A_* \in \mathcal{CP}(T(p+q)),\; B_* \in \mathcal{CP}(T(p+q)), \text{ and } U_* := A_* + B_* \in \mathcal{F}_*.$$

Since $A_* \in \mathcal{CP}(T(p+q))$, $B_* \in \mathcal{CP}(T(p+q))$, then the matrices A_* and B_* can be represented in the forms

$$A_* = \sum_{j=1}^{\bar{p}_*} \begin{pmatrix} \xi_1(j) \\ \xi_2(j) \end{pmatrix} (\xi_1(j)', \xi_2(j)'),\; B_* = \sum_{j=1}^{\bar{p}_*} \begin{pmatrix} \mu_1(j) \\ \mu_2(j) \end{pmatrix} (\mu_1(j)', \mu_2(j)'), \tag{4.6}$$

where $\bar{p}_* = (p+q)(p+q+1)/2$, $\xi_1(j) \in \mathbb{R}^p_+$, $\xi_2(j) \in \mathbb{R}^q_+$, $\mu_1(j) \in \mathbb{R}^p_+$, $\mu_2(j) \in \mathbb{R}^q_+$, for all $j = 1, ..., \bar{p}_*$. It follows from (4.5), (4.6), and equality $U_* = A_* + B_*$ that

$$\sum_{s\in\mathcal{S}} (\alpha_s\tau(s))(\alpha_s\tau(s))' = \sum_{j=1}^{\bar{p}_*} (\xi_1(j)\xi_1(j)' + \mu_1(j)\mu_1(j)'). \tag{4.7}$$

It is evident that

$$\sum_{s\in\mathcal{S}} (\alpha_s\tau(s))(\alpha_s\tau(s))' \in \mathcal{F},\; \sum_{j=1}^{\bar{p}_*} \xi_1(j)\xi_1(j)' \in \mathcal{CP}(T(p)),\; \sum_{j=1}^{\bar{p}_*} \mu_1(j)\mu_1(j)' \in \mathcal{CP}(T(p)).$$

Then it follows from (4.1) and (4.7) that

$$\xi_1(j)\xi_1(j)' \in \mathcal{F},\; \mu_1(j)\mu_1(j)' \in \mathcal{F} \;\forall j = 1, ..., \bar{p}_*.$$

Due to Lemma 3.2 these inclusions imply the equalities

$$\xi_1(j) = \beta_j\tau(s_j),\; \mu_1(j) = \bar{\beta}_j\tau(\bar{s}_j) \text{ with some } \beta_j \geqslant 0,\; s_j \in \mathcal{S},\; \bar{\beta}_j \geqslant 0,\; \bar{s}_j \in \mathcal{S},$$
$$\forall j = 1, ..., \bar{p}_*.$$

Consequently, for all $j = 1, ..., \bar{p}_*$,

$$\begin{pmatrix} \xi_1(j) \\ \xi_2(j) \end{pmatrix} = \begin{pmatrix} \beta_j\tau(s_j) \\ \xi_2(j) \end{pmatrix} = (\beta_j + ||\xi_2(j)||_1)\bar{t}(s_j),\; \bar{t}(s_j) \in T_*(s_j),$$

$$\begin{pmatrix} \mu_1(j) \\ \mu_2(j) \end{pmatrix} = \begin{pmatrix} \bar{\beta}_j\tau(\bar{s}_j) \\ \mu_2(j) \end{pmatrix} = (\bar{\beta}_j + ||\mu_2(j)||_1)\bar{t}(\bar{s}_j),\; \bar{t}(\bar{s}_j) \in T_*(\bar{s}_j).$$

These equalities and relations (4.4), (4.6) imply that $A_* \in \mathcal{F}_*$, $B_* \in \mathcal{F}_*$, and consequently $\mathcal{F}_*$ is a face of $\mathcal{CP}(T(p+q))$.

Now we will show that $\mathcal{F}_*$ is a non-exposed face of $\mathcal{CP}(T(p+q))$.

Let $\mathcal{F}^{exp}_* = \mathcal{CP}(T(p+q)) \cap X_*^{\perp}$ with some $X_* \in \mathcal{COP}(T(p+q))$ be the minimal exposed face of $\mathcal{CP}(T(p+q))$ containing $\mathcal{F}_*$. Hence

$$\bigcup_{s\in\mathcal{S}} T_*(s) \subset Z_0(X_*, T(p+q)) \implies \bar{\tau}(s) \in Z_0(X_*, T(p+q)) \;\forall s \in \mathcal{S}.$$

Let us rewrite the matrix X_* in the form

$$X_* = \begin{pmatrix} \tilde{X} & \tilde{X}_1 \\ \tilde{X}_1' & \tilde{X}_2 \end{pmatrix}, \text{ where } \tilde{X} \in \mathcal{COP}(T(p)).$$

Then

$$0 = (\bar{\tau}(s))' X_* \bar{\tau}(s) = (\tau(s))' \tilde{X} \tau(s) \ \forall s \in \mathcal{S}.$$

Taking into account these equalities, the inclusion $\tilde{X} \in \mathcal{COP}(T(p))$ and conditions (4.2), (4.3), we conclude that $\tau_*' \tilde{X} \tau_* = 0$. It follows from the latter equality that

$$\bar{\tau}_*' X_* \bar{\tau}_* = 0 \text{ with } \bar{\tau}_* = (\tau_*', 0, ..., 0)' \in \mathbb{R}^{p+q},$$

and consequently $\bar{\tau}_* \bar{\tau}_*' \in \mathcal{F}_*^{exp}$.

Let us show that $\bar{\tau}_* \bar{\tau}_*' \notin \mathcal{F}_*$. In fact, suppose the contrary: $\bar{\tau}_* \bar{\tau}_*' \in \mathcal{F}_*$. Then it follows from the latter inclusion that $\tau_* \tau_*' \in \mathcal{F}$. Hence taking into account this inclusion, Lemma 3.2 and equalities $||\tau_*||_1 = ||\tau(s)||_1 = 1$ for all $s \in \mathcal{S}$, we obtain that $\tau_* = \tau(s_*)$ with some $s_* \in \mathcal{S}$. But this contradicts the assumption that $\tau_* \in T(p) \setminus \{\tau(s), s \in \mathcal{S}\}$. Thus, we have proved that $\bar{\tau}_* \bar{\tau}_*' \notin \mathcal{F}_*$.

Since $\bar{\tau}_* \bar{\tau}_*' \notin \mathcal{F}_*$, $\bar{\tau}_* \bar{\tau}_*' \in \mathcal{F}_*^{exp}$, and $\mathcal{F}_*^{exp}$ is the minimal exposed face of $\mathcal{CP}(T(p+q))$ containing $\mathcal{F}_*$, we conclude that $\mathcal{F}_*$ is a non-exposed face of $\mathcal{CP}(T(p+q))$.

Now we will show that $\mathcal{F}_*$ is a non-polyhedral face. By construction, for any $s \in \mathcal{S}$, the set $T_*(s)$ contains a continuum of elements. Hence, the set $\mathcal{Z} := \bigcup_{s \in S} T_*(s)$ does not consist of a finite number of elements. Then it follows from Proposition 3.4 that the face $\mathcal{F}_*$ is not polyhedral. □

Corollary 4.1. *For any $p \geqslant 6$, there exist non-exposed non-polyhedral faces of $\mathcal{CP}(T(p))$.*

Proof. It was shown in [10] that for any $p \geqslant 5$, there exist non-exposed polyhedral faces of $\mathcal{CP}(T(p))$. Thus, it follows from this result and Theorem 4.1 that for any $p \geqslant 6$, there exist non-exposed non-polyhedral faces of $\mathcal{CP}(T(p))$. □

Remark 4.2. *It follows from the proof of Theorem* 4.1 *that statement of this theorem holds true, if $\mathcal{F}$ is a non-exposed (not necessary polyhedral) face of $\mathcal{CP}(T(p))$ and for $s \in \mathcal{S}$, the set $T_*(s)$ is as follows: $T_*(s) = \text{conv}\{\bar{\xi}(j), j \in \mathcal{J}(s), \ \mathbf{e}_{p+k}, k = 1, ..., q\}$, $\bar{\xi}(j) = (\xi'(j), 0, ..., 0)' \in \mathbb{R}^{p+q} \ \forall j \in \mathcal{J}(s)$, $\forall s \in \mathcal{S}$. Here we consider that the face $\mathcal{F}$ is represented in the form* (3.2) *where the set $Z_0(k_0)$ is as in* (3.10). *Note that such representation exists for any proper face of $\mathcal{CP}(T(p))$.*

5. Examples

In this section, we illustrate the application of Proposition 3.6 and Theorem 4.1 by examples.

We start with an example which illustrates Proposition 3.6.

Let us set $p = 6$ and consider the set $T(6) = \{t \in \mathbb{R}^6_+, \ ||t||_1 = 1\}$ and matrix

$$X_0 = \begin{pmatrix} H & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix} \in \mathbb{R}^{6\times 6}, \text{ where } H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

The matrix H is the Horn matrix and it is known (see, for example, [7]) that it is copositive, hence, $X_0 \in \mathcal{COP}(T(6))$.

Let us denote

$$\tau(1) = 0.5(1, 1, 0, 0, 0, 0)', \ \tau(2) = 0.5(0, 1, 1, 0, 0, 0)', \ \tau(3) = 0.5(0, 0, 1, 1, 0, 0)',$$
$$\tau(4) = 0.5(0, 0, 0, 1, 1, 0)', \ \tau(5) = 0.5(1, 0, 0, 0, 1, 0)', \ \tau(6) = (0, 0, 0, 0, 0, 1)'.$$

One can easily check that

$$Z_0(0) := Z_0(X_0, T(6)) = \text{conv}\{\tau(1), \tau(2), \tau(6)\} \cup \text{conv}\{\tau(2), \tau(3), \tau(6)\} \cup$$
$$\cup \text{conv}\{\tau(3), \tau(4), \tau(6)\} \cup \text{conv}\{\tau(4), \tau(5), \tau(6)\} \cup \text{conv}\{\tau(1), \tau(5), \tau(6)\}.$$

Let us consider a matrix

$$X_1 = \begin{pmatrix} D_0 & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix} \in \mathbb{R}^{6\times 6}, \text{ where } D_0 = \begin{pmatrix} 6.0 & -11.5 & 5.5 & 5.5 & -5.5 \\ -11.5 & 21.0 & -19.0 & 13.0 & 7.0 \\ 5.5 & -19.0 & 21.0 & -19.0 & 7.0 \\ 5.5 & 13.0 & -19.0 & 21.0 & -13.0 \\ -5.5 & 7.0 & 7.0 & -13.0 & 9.0 \end{pmatrix}.$$

Since for $t_* = (0.27, 0.46, 0.27, 0, 0, 0)' \in T(6)$ we have $t_*'X_1t_* = -0.3624$, it is evident that $X_1 \notin \mathcal{COP}(T(6))$.

Let us show that $X_1 \in \mathcal{COP}(Z_0(0))$. To do this we will show that $X_1 \in \mathcal{COP}(\text{conv}(Z_0(0))$ which is equivalent to $\tilde{D} := \mathcal{T}'X_1\mathcal{T} \in \mathcal{COP}(T(6))$, where

$$\mathcal{T} := (\tau(j), j = 1,...,6) \in \mathbb{R}^{6\times 6}.$$

In fact, the matrix $\tilde{D}$ takes the form

$$\tilde{D} = 4(X_0 + C), \text{ where } C = \begin{pmatrix} 0 & 0 & 0.25 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0.25 & 0 & 0 & 0 & 0.25 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since all elements of the matrix C are non-negative and $X_0 \in \mathcal{COP}(T(6))$, we conclude that $\tilde{D} \in \mathcal{COP}(T(6))$ and hence $X_1 \in \mathcal{COP}(\text{conv}(Z_0(0))$ that implies $X_1 \in \mathcal{COP}(Z_0(0))$.

Now let us find the set $Z_0(1) := Z_0(X_1, Z_0(0))$. It is easily to do this via the matrix $\tilde{D}$. It is easy to see that

$$Z_0(\tilde{D}, T(6)) = \text{conv}\{\tau(1), \tau(6)\} \cup \text{conv}\{\tau(2), \tau(6)\} \cup \text{conv}\{\tau(3), \tau(6)\} \cup$$
$$\cup \text{conv}\{\tau(4), \tau(6)\} \cup \text{conv}\{\tau(5), \tau(6)\}.$$

This implies that

$$Z_0(X_1, \text{conv}(Z_0(0))) = \text{conv}\{\xi(1), \xi(6)\} \cup \text{conv}\{\xi(2), \xi(6)\} \cup \text{conv}\{\xi(3), \xi(6)\} \cup$$
$$\cup \text{conv}\{\xi(4), \xi(6)\} \cup \text{conv}\{\xi(5), \xi(6)\},$$

where

$$\xi(1) = 0.5(\tau(1) + \tau(2)),\ \xi(2) = 0.5(\tau(2) + \tau(3)),\ \xi(3) = 0.5(\tau(3) + \tau(4)),$$

$$\xi(4) = 0.5(\tau(4) + \tau(5)),\ \xi(5) = 0.5(\tau(1) + \tau(5)),\ \xi(6) = \tau(6).$$

It is easy to see that if $T_1 \subset T_2 \subset T$ then $D \in \mathcal{COP}(T_2)$ implies that $D \in \mathcal{COP}(T_1)$, and $Z_0(D, T_1) \subset \subset Z_0(D, T_2)$. Hence, if $Z_0(D, T_2) \subset T_1$, then $Z_0(D, T_1) = Z_0(D, T_2)$.

In our case, we have $T_2 := \text{conv}(Z_0(0))$, $T_1 := Z_0(0)$, $X_1 \in \mathcal{COP}(T_2)$ and $Z_0(X_1, T_2) \subset T_1$. Hence, the equality $Z_0(X_1, T_1) = Z_0(X_1, T_2)$ holds true, and we obtain

$$Z_0(1) := Z_0(X_1, Z_0(0)) = Z_0(X_1, \text{conv}(Z_0(0))).$$

Consequently,

$$\mathcal{S} = \{1,...,5\},\ \mathcal{J}(s) = \{s, 6\}\ \forall s \in \mathcal{S};,\ \mathcal{J} = \{1,...,6\},$$
$$M_*(1) = \{1,2,3,6\},\ M_*(2) = \{2,3,4,6\},\ M_*(3) = \{3,4,5,6\},$$
$$M_*(4) = \{1,4,5,6\},\ M_*(5) = \{1,2,5,6\},\ M_*(6) = \{1,...,6\}.$$

Consider the set

$$\mathcal{F} := \{U = \sum_{i=1}^{p_*} \alpha_i t(i) t(i)',\ \alpha_i \geqslant 0,\ t(i) \in Z_0(1)\}, p_* = p(p+1)/2.$$

Since matrices X_0, X_1 and the sets $Z_0(0)$, $Z_0(1)$ satisfy conditions (3.1) with $k_0 = 1$, we conclude that $\mathcal{F}$ is a face of $\mathcal{CP}(T(6))$.

Let us show that the face $\mathcal{F}$ is non-exposed. To do this we will use Proposition 3.6.

Set $t^* = \tau(1) \in Z_0(0) \setminus Z_0(1)$ and consider the sets of matrices

$$\mathcal{B} := \{D \in \mathbb{S}(6) : e_k' D\xi(j) = 0 \ \forall k \in M_*(j), \ \forall j \in \mathcal{J}\},$$
$$\mathcal{W} := \{D \in \mathbb{S}(6) : e_k' Dt^* = 0 \ \forall k \in \mathrm{supp}(t^*)\} \text{ with } \mathrm{supp}(t^*) = \{1,2\},$$

where $\{e_k, k = 1,...,6\}$ is the standard basis in $\mathbb{R}^6$.

For any matrix $D \in \mathbb{S}(6)$ with elements $d_{ij}, i = 1,...,6, j = i,...,6$, we define the vector $\mathrm{svec}(D) \in \mathbb{R}^{21}$ by the rule (see [15])

$$\mathrm{svec}(D) = (d_{11}, \sqrt{2}d_{12}, ..., \sqrt{2}d_{16}, d_{22}, \sqrt{2}d_{23}, ..., \sqrt{2}d_{26}, ..., d_{66})^\top.$$

It is easy to construct matrices $\mathbb{B} \in \mathbb{R}^{26\times 21}$, $\mathbb{W} \in \mathbb{R}^{2\times 21}$ such that

$$\mathcal{B} = \{D \in \mathbb{S}(6) : \mathbb{B}\,\mathrm{svec}(D) = \mathbf{0}\}, \quad \mathcal{W} = \{D \in \mathbb{S}(6) : \mathbb{W}\,\mathrm{svec}(D) = \mathbf{0}\}.$$

In our example, we have $\mathrm{rank}\,\mathbb{B} = 20$ and $\mathrm{rank}\begin{pmatrix}\mathbb{B}\\ \mathbb{W}\end{pmatrix} = 20$. Hence it follows from (3.17) and Proposition 3.6 that the face $\mathcal{F}$ is non-exposed.

Since the set $Z_0(k_0) = Z_0(1) = Z_0(X_1, \mathrm{conv}(Z_0(0))$ contains a continuum of elements, it follows from Proposition 3.4 that $\mathcal{F}$ is non-polyhedral.

In [10], for any $p \geqslant 5$, examples of non-exposed polyhedral faces of $\mathcal{CP}(T(p))$ were given. Hence, to illustrate Theorem 4.1 we can choose any non-exposed face form [10]. To simplify the presentation, let us choose $p = 5$. Set $\mathcal{S} = \{1,...,5\}$,

$$\tau(1) = (1,2,1,0,0)'/4, \ \tau(2) = (0,1,2,1,0)'/4, \ \tau(3) = (0,0,1,2,1)'/4,$$
$$\tau(4) = (1,0,0,1,2)'/4, \ \tau(5) = (2,1,0,0,1)'/4.$$

It was shown in [9; 10] that the set $\mathcal{F} := \mathrm{cone}\{\tau(s), i \in \mathcal{S}\}$ is a non-exposed face of $\mathcal{CP}(5)$. Let us choose $q = 2$ and denote

$$\bar\tau(1) = (1,2,1,0,0,0,0)'/4, \ \bar\tau(2) = (0,1,2,1,0,0,0)'/4, \ \bar\tau(3) = (0,0,1,2,1,0,0)'/4,$$
$$\bar\tau(4) = (1,0,0,1,2,0,0)'/4, \ \bar\tau(5) = (2,1,0,0,1,0,0)'/4,$$

$$T_*(s) = \mathrm{conv}\{\bar\tau(s), e_6, e_7\} = \{(\alpha_1\tau'(s), \alpha_2, \alpha_3)', \alpha_i \geqslant 0, i = 1,2,3; \sum_{i=1}^{3} \alpha_i = 1\} \subset \mathbb{R}^7_+, \ s \in \mathcal{S}.$$

Then it follows from Theorem 4.1 that the set $\mathcal{F}_* := \mathrm{cone}\{\bar t \bar t', \bar t \in \bigcup_{s\in\mathcal{S}} T_*(s)\}$ is a non-exposed non-polyhedral face of $\mathcal{CP}(7)$.

Conclusion. In this paper, we studied properties of non-exposed faces of the cone of completely positive matrices. A special attention was paid on non-polyhedral faces. Based on results obtained, we proved easily tested sufficient conditions for a face to be non-exposed one. The novelty of the results obtained lies in the fact that it was proved and illustrated that for any $p \geqslant 6$, there exist non-polyhedral non-exposed faces of the cone of completely positive $p \times p$ matrices. This makes a significant contribution to the study of the facial structure of the completely positive cone.

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