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PARAMETRIC SEMI-INFINITE PROGRAMMING PROBLEMS: PROPERTIES OF SOLUTION FUNCTIONS

O. I. Kostyukova[1], **T. V. Tchemisova**[2]

[1]*Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus*
[2]*Department of Mathematics, University of Aveiro, Campus Universitario Santiago, Aveiro, Portugal*
e-mail:kostyukova@im.bas-net.by, tatiana@ua.pt

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Abstract. We study sensitivity and asymptotic behavior of solutions to parametric convex semi-infinite programming problems with compact polyhedral index sets. We analyze properties of optimal solutions, Lagrange multipliers, and active sets as the parameter approaches a nominal value. Under standard smoothness, convexity, and constraint qualification assumptions, we characterize the limits and first-order derivatives of the associated primal and dual mappings.

ПАРАМЕТРИЧЕСКИЕ ЗАДАЧИ ПОЛУБЕСКОНЕЧНОГО ПРОГРАММИРОВАНИЯ: СВОЙСТВА ФУНКЦИЙ РЕШЕНИЙ

О. И. Костюкова[1], **Т. В. Чемисова**[2]

[1]*Институт математики НАН Беларуси, Минск, Беларусь*
[2]*Факультет математики, Университет Авейру, Университетский кампус Сантьяго, Авейру, Португалия*
e-mail: kostyukova@im.bas-net.by, tatiana@ua.pt

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Ключевые слова: выпуклая оптимизация, полубесконечное программирование, параметрические задачи, прямая и двойственная задачи, функции решений

Аннотация. В работе исследуются анализ чувствительности и асимптотическое поведение решений параметрических выпуклых задач полубесконечного программирования с компактными полиэдральными индексными множествами. Анализируются свойства оптимальных решений, множителей Лагранжа и множества активных индексов при стремлении параметра к номинальному значению. При стандартных предположениях о гладкости, выпуклости и выполнении условий регулярности ограничений получены соотношения, характеризующие пределы и производные первого порядка соответствующих прямых и двойственных функций решений.

1. Introduction

Parametric optimization, which investigates how the optimal solutions and value functions of optimization problems respond to perturbations in problem data, constitutes a vital and rapidly evolving field of mathematical programming [1–3]. Understanding the sensitivity, stability, and well-posedness of optimization models is crucial for both theoretical analysis and practical applications, such as engineering design, economic forecasting, and robust decision-making under uncertainty [4–8].

Over the past few decades, substantial research efforts have been dedicated to analyzing parametric optimization problems. The existing literature can be broadly classified into three main research directions, according to the structure of the underlying optimization models:

A. **Parametric Nonlinear Programming (NLP)**, where the problems involve a finite number of variables and constraints. Classic foundations and comprehensive surveys in this area can be found in the pioneering works of Fiacco [1], Bank et al. [9], see also [2; 10–12].

B. **Parametric Semi-Infinite Programming (SIP)**, which deals with a finite number of variables but infinitely many constraints. Key contributions to the theoretical foundations, stability, and sensitivity analysis of SIP include the landmark studies by Hettich and Kortanek [4], López and Still [13], Gobera and López [14], as well by the authors of [15; 16].

C. **General Parametric Optimization in Banach Spaces**, which provides an abstract and unified framework for the analysis of optimization problems in infinite-dimensional settings. Representative contributions in this area include the monographs by Bonnans and Shapiro [3] and Rockafellar and Wets [17], see also [18; 19].

A fundamental concept that connects convergence and stability across these areas is well-posedness under perturbations. This notion, extensively developed by Levitin and Polyak [20; 21], has become a cornerstone of modern sensitivity and error bound analysis [22].

Formally, three classes of models introduced above satisfy the inclusion relation $A \subset B \subset C$, where SIP occupies a unique intermediate position. Due to the presence of infinitely many constraints, standard results and techniques from finite-dimensional parametric NLP (Group A) cannot be directly applied to SIP problems without facing significant topological and analytical challenges. On the other hand, since SIP constitutes a particular class of optimization problems in Banach spaces (Group C), the abstract perturbation framework developed for general infinite-dimensional optimization remains applicable.

However, relying solely on the general framework of Group C often yields conditions that are overly abstract, non-constructive, and difficult to verify in practice. By exploiting the specific structure of SIP – namely, a finite-dimensional decision space combined with an infinite constraint system indexed by a compact set – these general results can be significantly refined and simplified, making them more transparent, computationally tractable, and readily verifiable. Furthermore, the explicit geometric and analytical structure of SIP allows one to establish stronger, sharper, and more exact sensitivity results than those obtained by a direct application of the general infinite-dimensional theory. Consequently, SIP occupies a particularly attractive position, combining the rich theoretical framework of infinite-dimensional optimization with structural properties that permits a more precise and practical analysis.

Interest in this area has experienced a strong resurgence in recent years. Current research (2020–2026) has significantly advanced the study of parametric and stability properties of semi-infinite problems, extending classical results to nonconvex SIP, generalized SIP (GSIP), and robust optimization frameworks. These developments are reflected in recent surveys and research contributions by Mitsos et al. [23], López [24], as well as in a growing number of engineering and applied optimization studies [5].

In this paper, we consider the following parametric convex semi-infinite programming problem

$$\begin{aligned} (\mathrm{SIP}(\varepsilon)): \qquad & \min_{x} \quad f_*(x,\varepsilon), \\ & \text{s.t.} \quad f(x,t,\varepsilon) \leqslant 0, \quad \forall t \in T := \{t \in \mathbb{R}^p : a_i^\top t \leqslant b_i,\ i \in I\}, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the decision variable; $t \in \mathbb{R}^p$ is the constraint index, and $\varepsilon \in \mathbb{R}$ is the parameter. The index set T is a non-empty compact polyhedron defined by a finite number of linear inequalities, vectors $a_i \in \mathbb{R}^p$, $i \in I$, and numbers b_i, $i \in I$, are supposed to be given. We assume that the scalar functions $f_*(x,\varepsilon)$ and $f(x,t,\varepsilon)$ are sufficiently smooth with respect to all their arguments. Furthermore, they are assumed to be convex with respect to $x \in \mathbb{R}^n$ for all $t \in T$ and all $\varepsilon \in \mathcal{E}(\varepsilon_0) := [\varepsilon_0 - \delta, \varepsilon_0 + \delta]$, where $\delta > 0$ is a sufficiently small positive number. Here and in what follows, the symbol $^\top$ stands for the transpose operation.

For a fixed parameter value ε, let $x(\varepsilon)$ be an optimal solution to problem $(\mathrm{SIP}(\varepsilon))$, and let $T_a(x(\varepsilon),\varepsilon)$ denote the corresponding active index set defined as:

$$T_a(x(\varepsilon),\varepsilon) := \{t \in T : f(x(\varepsilon),t,\varepsilon) = 0\}.$$

Suppose that the active set above consists of a finite number of elements, i. e., $T_a(x(\varepsilon),\varepsilon) = \{t_s(\varepsilon),\ s \in S(\varepsilon)\}$, with $|S(\varepsilon)| < \infty$. Then, under standard constraint qualification assumptions (such as the Slater constraint qualification), the first-order optimality conditions guarantee the existence of Lagrange multipliers $y_s(\varepsilon) \geqslant 0$, $s \in S(\varepsilon)$, such that:

$$\frac{\partial f_*(x(\varepsilon),\varepsilon)}{\partial x} + \sum_{s \in S(\varepsilon)} y_s(\varepsilon) \frac{\partial f(x(\varepsilon),t_s(\varepsilon),\varepsilon)}{\partial x} = 0.$$

The main objective of this paper is to systematically study the properties and asymptotic behavior of the mappings representing the primal solution $x(\varepsilon)$, the dual solution $y(\varepsilon) = (y_s(\varepsilon),\ s \in S(\varepsilon))$, and the active indices $t_s(\varepsilon)$, $s \in S(\varepsilon)$, as the parameter ε approaches the unperturbed value ε_0. Specifically, we analyze the limiting values of these mappings and their derivatives at $\varepsilon = \varepsilon_0$. We show that both the limiting values and the derivatives can be characterized as solutions of certain linearly constrained quadratic optimization problems. This characterization enables us to derive a system of linear equations and inequalities that must be satisfied by the corresponding limiting values and derivatives.

The theoretical results obtained in this paper are intended to provide a solid foundation for future research on the corresponding inverse problem. Specifically, given a known solution to the unperturbed problem (SIP(ε_0)), our goal is to predict and accurately characterize the local behavior of the primal-dual solution trajectories under small perturbations of the parameter ε. In particular, we seek to describe and track the evolution of both primal and dual solutions as ε varies in a neighborhood of ε_0.

2. Problem Statement, Notation, and Assumptions

In this paper, we consider the parametric semi-infinite programming problem (SIP(ε)) formulated in the Introduction. For simplicity and without loss of generality, we suppose hereinafter that $\varepsilon_0 = 0$.

We also assume that the following standard assumption is fulfilled.

Assumption 2.1. *Let the following conditions be satisfied:*

1. *The constraints of problem* (SIP(0)) *satisfy the Slater condition, i. e., there exists* $\bar{x} \in \mathbb{R}^n$ *such that* $f(\bar{x}, t, 0) < 0$ *for all* $t \in T$.

2. *The sets of feasible solutions* $X(\varepsilon) := \{x \in \mathbb{R}^n : f(x,t,\varepsilon) \leqslant 0,\ t \in T\}$ *of parametric problems* (SIP(ε))*, are non-empty and uniformly bounded for* $\varepsilon \in \mathcal{E}(0) := [-\delta, \delta]$*, for a sufficiently small* $\delta > 0$.

Under Assumption 2.1, problem (SIP(ε)) has an optimal solution for any $\varepsilon \in \mathcal{E}(0)$ (see [3]). Let $x(\varepsilon)$ be an optimal solution to (SIP(ε)).

Assumption 2.2. *Suppose that for all* $\varepsilon \in \mathcal{E}(0)$*, the set* $T_a(x(\varepsilon), \varepsilon)$ *consists of a finite number of elements, i. e. it admits a representation*

$$T_a(x(\varepsilon), \varepsilon) = \{t_s(\varepsilon),\ s \in S(\varepsilon)\}$$

and there exists $M > 0$ *such that the cardinality of the set* $S(\varepsilon)$ *is less than* M.

Consider $\varepsilon \in \mathcal{E}(0)$. It is evident that for all $s \in S(\varepsilon)$, the vector $t_s(\varepsilon)$ is a global optimal solution of the lower level problem (LLP($x(\varepsilon), \varepsilon$)):

$$(\mathrm{LLP}(x, \varepsilon)): \qquad \max_t \quad f(x,t,\varepsilon),\ \text{ s.t. } \quad t \in T.$$

For $s \in S(\varepsilon)$, denote

$$I_s(\varepsilon) := I_a(t_s(\varepsilon)),\ \text{where}\ I_a(t) := \{i \in I : a_i^\top t = b_i\}.$$

Then, for $s \in S(\varepsilon)$, according to the optimality conditions for $t_s(\varepsilon)$ in problem (LLP($x(\varepsilon), \varepsilon$)), there exists a vector $\mu(s, \varepsilon) = (\mu_i(s, \varepsilon),\ i \in I_s(\varepsilon))^\top$ such that:

$$\frac{\partial f(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial t} = \sum_{i \in I_s(\varepsilon)} a_i \mu_i(s, \varepsilon), \quad \mu_i(s, \varepsilon) \geqslant 0,\ i \in I_s(\varepsilon), \tag{1}$$

$$l^\top \frac{\partial^2 f(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial t^2} l \leqslant 0, \quad \forall l \in K(t_s(\varepsilon), \varepsilon),$$

where

$$K(t, \varepsilon) := \left\{ l \in \mathbb{R}^p : a_i^\top l \leqslant 0,\ i \in I_a(t),\ l^\top \frac{\partial f(x(\varepsilon), t, \varepsilon)}{\partial t} \geqslant 0 \right\}.$$

Note that the set $K(t_s(\varepsilon), \varepsilon)$ admits the following representation:

$$K(t_s(\varepsilon), \varepsilon) = \{l \in \mathbb{R}^p : a_i^\top l = 0 \text{ if } \mu_i(s, \varepsilon) > 0,\ a_i^\top l \leqslant 0 \text{ if } \mu_i(s, \varepsilon) = 0,\ i \in I_s(\varepsilon)\}.$$

Under the Slater condition, the first-order optimality conditions for $x(\varepsilon)$ in problem (SIP(ε)) imply the existence of Lagrange multipliers $y_s(\varepsilon)$, $s \in S(\varepsilon)$, such that

$$\frac{\partial f_*(x(\varepsilon),\varepsilon)}{\partial x} + \sum_{s\in S(\varepsilon)} y_s(\varepsilon)\frac{\partial f(x(\varepsilon),t_s(\varepsilon),\varepsilon)}{\partial x} = 0, \quad y_s(\varepsilon) \geqslant 0,\ s \in S(\varepsilon). \tag{2}$$

Assumptions 2.1, 2.2 and the compactness of the set T imply the existence of an index set $\bar{J}$ and a subsequence ε_m, $m = 1,2,\ldots$, such that $\varepsilon_m \to 0$ as $m \to \infty$,

$$S(\varepsilon_m) = S \text{ for all sufficiently large } m, \quad S = \bigcup_{j\in\bar{J}} S_j, \quad S_j \cap S_i = \varnothing \text{ if } i \neq j,$$

and there exist limits

$$\lim_{m\to\infty} t_s(\varepsilon_m) =: t_j^*,\ s \in S_j,\ j \in \bar{J}, \tag{3}$$

$$\lim_{m\to\infty} y_s(\varepsilon_m) =: y_s^*,\ s \in S, \quad \lim_{m\to\infty} x(\varepsilon_m) =: x^0. \tag{4}$$

In what follows, we simplify the notation by considering the sequence $\varepsilon \to 0$ instead of the subsequence $\varepsilon_m \to 0$ as $m \to \infty$, without loss of generality.

Denote

$$T_a(x^0,0) = \{t_j^0,\ j \in J^0\},$$

where $J^0 := S(0)$. It follows from Assumption 2.2 that $|J^0| < \infty$, and it is evident that $\{t_j^*,\ j \in \bar{J}\} \subset T_a(x^0,0)$, where t_j^*, $j \in \bar{J}$, and x^0 are defined in (3) and (4). Hence, we can assume that

$$t_j^0 = t_j^*, \quad j \in \bar{J} \subset J^0.$$

Let us set

$$y_j^0 := \sum_{s\in S_j} y_s^*,\ j \in \bar{J}; \quad y_j^0 = 0,\ j \in J^0 \setminus \bar{J}. \tag{5}$$

It is easy to show that the vector x^0 is a feasible solution to unperturbed problem (SIP(0)) and that the following conditions are satisfied:

$$\frac{\partial f_*(x^0,0)}{\partial x} + \sum_{j\in J^0} y_j^0 \frac{\partial f(x^0,t_j^0,0)}{\partial x} = 0, \quad y_j^0 \geqslant 0,\ j \in J^0. \tag{6}$$

These conditions imply the optimality of x^0 in problem (SIP(0)), taking into account its convexity.

By definition, $T_a(x^0,0) = T_a(x(\varepsilon),\varepsilon)|_{\varepsilon=0}$, hence, as mentioned above, the vectors t_j^0, $j \in J^0$, are global optimal solutions to the lower level problem (LLP($x^0,0$)). Consequently, there exist vectors

$$\mu(j) = (\mu_i(j),\ i \in I_j)^\top, \quad j \in J^0,$$

such that

$$\frac{\partial f(x^0,t_j^0,0)}{\partial t} = \sum_{i\in I_j} a_i \mu_i(j), \quad \mu_i(j) \geqslant 0,\ i \in I_j,\ j \in J^0, \tag{7}$$

$$l^\top \frac{\partial^2 f(x^0,t_j^0,0)}{\partial t^2} l \leqslant 0, \quad \forall l \in K(j), \tag{8}$$

where

$$I_j := \{i \in I : a_i^\top t_j^0 = b_i\},$$

$$K(j) := \{l \in \mathbb{R}^p : a_i^\top l = 0 \text{ if } \mu_i(j) > 0,\ a_i^\top l \leqslant 0 \text{ if } \mu_i(j) = 0,\ i \in I_j\} =$$

$$= \{l \in \mathbb{R}^p : a_i^\top l \leqslant 0,\ i \in I_j;\ l^\top \frac{\partial f(x^0,t_j^0,0)}{\partial t} = 0\}, \quad j \in J^0.$$

Assumption 2.3. *In the terminology defined earlier, there exist finite limits:*

$$\lim_{\varepsilon\to 0}\frac{x(\varepsilon)-x^0}{\varepsilon}=:\varkappa^*, \tag{9}$$

$$\lim_{\varepsilon\to 0}\frac{t_s(\varepsilon)-t_j^0}{\varepsilon}=:\tau_s^*,\quad s\in S_j,\ j\in\bar{J}. \tag{10}$$

For simplicity, in what follows we use the following shorthand notation:

$$f_j(\cdot):=f(x^0,t_j^0,0),\quad \frac{\partial^{k+s}f_j(\cdot)}{\partial x^k\partial y^s}:=\frac{\partial^{k+s}f(x^0,t_j^0,0)}{\partial x^k\partial y^s},\quad j\in J^0,\ s\geqslant 0,\ k\geqslant 0,\ k+s\leqslant 2.$$

Let us denote

$$\begin{gathered}\bar{L}(x,\varepsilon):=f_*(x,\varepsilon)+\sum_{j\in J^0}y_j^0f(x,t_j^0,\varepsilon);\ D:=\frac{\partial^2\bar{L}(x^0,0)}{\partial x^2},\quad c:=\frac{\partial^2\bar{L}(x^0,0)}{\partial x\partial\varepsilon};\\ D_j:=\frac{\partial^2f_j(\cdot)}{\partial t^2},\ A_j:=\frac{\partial^2f_j(\cdot)}{\partial t\partial x},\ c_j:=\frac{\partial^2f_j(\cdot)}{\partial t\partial\varepsilon},\ q_j:=\frac{\partial f_j(\cdot)}{\partial x},\quad \omega_j=\frac{\partial f_j(\cdot)}{\partial\varepsilon},\ j\in J^0.\end{gathered} \tag{11}$$

Consider the following linear programming problem:

$$(\mathrm{LP}):\quad \max\sum_{j\in J^0}y_j\omega_j,\quad \text{s. t.}\quad \frac{\partial f_*(x^0,0)}{\partial x}+\sum_{j\in J^0}y_jq_j=0,\quad y_j\geqslant 0,\ j\in J^0,$$

and its dual problem:

$$(\mathrm{DLP}):\qquad \min\qquad \varkappa^\top\frac{\partial f_*(x^0,0)}{\partial x},\quad \text{s. t.}\quad q_j^\top\varkappa+\omega_j\leqslant 0,\ j\in J^0.$$

It follows from (6) that the vector y^0 is a feasible solution to problem (LP). Moreover, by the Slater condition (see Assumption 2.1), the feasible set of this problem is bounded. Hence, the optimal solution set of problem (LP) is nonempty and also bounded. Consequently, both problems (LP) and (DLP) have optimal solutions.

Let Y^0 be the set of all optimal solutions to problem (LP). Introduce the following sets:

$$J_*^*:=\{j\in J^0:\exists y\in Y^0 \text{ such that } y_j>0\},\quad J_0^*:=J^0\setminus J_*^*.$$

It is evident that there exists $\bar{y}\in Y^0$ such that

$$\bar{y}_j>0\quad\forall j\in J_*^*,\quad \bar{y}_j=0\quad\forall j\in J_0^*. \tag{12}$$

Then, it is easy to show that the optimal solution set $\mathcal{X}^*$ of problem (DLP) can be characterized as follows:

$$\mathcal{X}^*=\{\varkappa\in\mathbb{R}^n:q_j^\top\varkappa+\omega_j=0,\ j\in J_*^*,\quad q_j^\top\varkappa+\omega_j\leqslant 0,\ j\in J_0^*\}. \tag{13}$$

3. Main results

In this section, we will use the following functions:

$$\Psi_j(\tau,x):=\frac{1}{2}\tau^\top D_j\tau+c_j^\top\tau+x^\top A_j\tau,\quad x\in\mathbb{R}^n,\ \tau\in\mathbb{R}^p,\ j\in J^0; \tag{14}$$

$$\Omega_j(\varkappa,\rho):=\frac{1}{2}\varkappa^\top\frac{\partial^2f_j(\cdot)}{\partial x^2}\varkappa+\frac{\partial^2f_j(\cdot)}{\partial x\partial\varepsilon}\varkappa+\frac{1}{2}\frac{\partial^2f_j(\cdot)}{\partial\varepsilon^2}+\rho,\quad \varkappa\in\mathbb{R}^n,\ \rho\in\mathbb{R}; \tag{15}$$

$$\Omega_0(\varkappa):=\frac{1}{2}\varkappa^\top\frac{\partial^2f_*(x^0,0)}{\partial x^2}\varkappa+\frac{\partial^2f_*(x^0,0)}{\partial x\partial\varepsilon}\varkappa+\frac{1}{2}\frac{\partial^2f_*(x^0,0)}{\partial\varepsilon^2},\quad \varkappa\in\mathbb{R}^n;$$

$$\rho_j(\varkappa):=\max_{\tau\in K(j)}\Psi_j(\tau,\varkappa),\quad j\in J_*^*; \tag{16}$$

$$\Delta x(\varepsilon)=x(\varepsilon)-x^0,\quad \Delta t_s(\varepsilon)=t_s(\varepsilon)-t_j^0,\quad s\in S_j,\ j\in\bar{J}.$$

Notice that functions $\Omega_j(\varkappa,\rho)$, $j\in J_*^*$, and $\Omega_0(\varkappa)$ are convex with respect to their variables, functions $\Psi_j(\tau,x)$, $j\in J^0$, are not convex with respect to $\tau\in\mathbb{R}^p$ in general.

3.1. Properties of vector y^0 and derivatives τ_s^*, $s \in S_j$, $j \in \bar{J}$

In this subsection, we show that the vector y^0 defined in (5) is a solution to certain linear programming problems, and that for $j \in \bar{J}$, the vectors τ_s^*, $s \in S_j$, solve a non-convex linear-quadratic problem.

Theorem 3.1. *Let Assumptions* 2.1–2.3 *hold. Consider the vector* $y^0 = (y_j^0,\ j \in J^0)$ *with entries defined in* (5)*, and the vectors* $\varkappa^*$ *and* τ_s^*, $s \in S_j$, $j \in \bar{J}$, *defined in* (9) *and* (10)*, respectively. Then:*

i) the vectors $y^0 = (y_j^0,\ j \in J^0)$ *and* $\varkappa^*$ *are optimal solutions to problems* (LP) *and* (DLP)*;*

ii) for $j \in \bar{J}$, *the vectors* τ_s^*, $s \in S_j$, *solve the corresponding problem*

$$\max \Psi_j(\tau, \varkappa^*), \quad s.\,t. \quad \tau \in K(j). \tag{17}$$

Proof. Let us prove statement i). To this end, we first show that for $j \in \bar{J}$, the inclusions $\tau_s^* \in K(j)$, $s \in S_j$, hold true.

It is evident that $I_s(\varepsilon) = I_a(t_s(\varepsilon)) \subset I_j$ for sufficiently small $\varepsilon > 0$. Hence, for such ε and $\Delta t_s(\varepsilon) := t_s(\varepsilon) - t_j^*$, we have

$$a_i^\top \Delta t_s(\varepsilon) = 0, \quad i \in I_s(\varepsilon), \quad a_i^\top \Delta t_s(\varepsilon) \leqslant 0, \quad i \in I_j \setminus I_s(\varepsilon). \tag{18}$$

Dividing both sides of (18) by ε and passing to the limit yields:

$$a_i^\top \tau_s^* \leqslant 0, \quad i \in I_j,\ s \in S_j,\ j \in \bar{J}.$$

Using (18), we multiply both sides of (1) by $\Delta t_s(\varepsilon)$ to get:

$$\frac{\partial f^\top(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial t} \Delta t_s(\varepsilon) = \sum_{i \in I_s(\varepsilon)} a_i^\top \Delta t_s(\varepsilon) \mu_i(s, \varepsilon) = 0.$$

Dividing both sides of the last equality by ε and passing to the limit, we obtain

$$\frac{\partial f_j^\top(\cdot)}{\partial t} \tau_s^* = 0, \quad s \in S_j,\ j \in \bar{J}. \tag{19}$$

Hence, we have shown that $\tau_s^* \in K(j)$, $s \in S_j$, $j \in \bar{J}$.

By construction,

$$f(x^0, t_j^0, 0) = 0, \quad j \in J^0,$$
$$f(x(\varepsilon), t_s(\varepsilon), \varepsilon) = 0, \quad s \in S_j,\ j \in \bar{J}; \quad f(x(\varepsilon), t_j^0, \varepsilon) \leqslant 0, \quad j \in J^0 \setminus \bar{J}.$$

Hence,

$$0 = f(x(\varepsilon), t_s(\varepsilon), \varepsilon) =$$
$$= \frac{\partial f_j^\top(\cdot)}{\partial x} \Delta x(\varepsilon) + \frac{\partial f_j^\top(\cdot)}{\partial t} \Delta t_s(\varepsilon) + \frac{\partial f_j(\cdot)}{\partial \varepsilon} \varepsilon + o(\| (\Delta x(\varepsilon), \Delta t_s(\varepsilon), \varepsilon) \|),\ s \in S_j,\ j \in \bar{J}; \tag{20}$$
$$f(x(\varepsilon), t_j^0, \varepsilon) = \frac{\partial f_j^\top(\cdot)}{\partial x} \Delta x(\varepsilon) + \frac{\partial f_j(\cdot)}{\partial \varepsilon} \varepsilon + o(\| (\Delta x(\varepsilon), \varepsilon) \|) \leqslant 0, \quad j \in J^0 \setminus \bar{J}. \tag{21}$$

After dividing both sides of (20) and (21) by ε and letting $\varepsilon \to 0$, we obtain

$$\frac{\partial f_j^\top(\cdot)}{\partial x} \varkappa^* + \frac{\partial f_j^\top(\cdot)}{\partial t} \tau_s^* + \frac{\partial f_j(\cdot)}{\partial \varepsilon} = 0, \quad s \in S_j,\ j \in \bar{J},$$
$$\frac{\partial f_j^\top(\cdot)}{\partial x} \varkappa^* + \frac{\partial f_j(\cdot)}{\partial \varepsilon} \leqslant 0, \quad j \in J^0 \setminus \bar{J}.$$

Taking into account (19), the above relations imply:

$$q_j^\top \varkappa^* + \omega_j = 0, \quad j \in \bar{J}, \quad q_j^\top \varkappa^* + \omega_j \leqslant 0, \quad j \in J^0 \setminus \bar{J}. \tag{22}$$

Hence the vector $\varkappa^*$ is a feasible solution to problem (DLP).

Consider the vector $y^0 = (y_j^0,\ j \in J^0)^\top$, where the components y_j^0, $j \in J^0$, are defined in (5). It follows from (6) that y^0 is a feasible solution to problem (LP). Moreover, the relations (22) imply that y^0 is an optimal solution to (LP) and $\varkappa^*$ is the corresponding dual optimal solution; that is, $\varkappa^*$ solves the problem (DLP). Hence, statement i) is proved.

Now, let us prove statement ii) by showing that for $j \in \bar{J}$, the vectors τ_s^*, $s \in S_j$, are global optimal solutions to problem (17).

Suppose, to the contrary, that for some $j \in \bar{J}$ and $s \in S_j$, there exists a vector $\tau_j \in K(j)$ such that

$$-\frac{1}{2}\tau_j^\top D_j \tau_j - (c_j + A_j^\top \varkappa^*)^\top \tau_j < -\frac{1}{2}\tau_s^{*\top} D_j \tau_s^* - (c_j + A_j^\top \varkappa^*)^\top \tau_s^*. \tag{23}$$

Since $\tau_j \in K(j)$, there exists $\varepsilon_* > 0$ such that $t_j^0 + \varepsilon\tau_j \in T$ for all $0 \leqslant \varepsilon \leqslant \varepsilon_*$.
As mentioned above, for any $s \in S_j$ and $\varepsilon \in \mathcal{E}(0)$, we have

$$0 = f(x(\varepsilon), t_s(\varepsilon), \varepsilon) \geqslant f(x(\varepsilon), t, \varepsilon), \quad \forall t \in T. \tag{24}$$

Computing the Taylor expansions yields:

$$\begin{aligned} 0 = f(x(\varepsilon), t_s(\varepsilon), \varepsilon) = f_j(\cdot) + \frac{\partial f_j^\top(\cdot)}{\partial x}\Delta x(\varepsilon) + \frac{\partial f_j^\top(\cdot)}{\partial t}\Delta t_s(\varepsilon) + \frac{\partial f_j(\cdot)}{\partial \varepsilon}\varepsilon + \\ + \tfrac{1}{2}\Delta x^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial x^2}\Delta x(\varepsilon) + \tfrac{1}{2}\Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t^2}\Delta t_s(\varepsilon) + \tfrac{1}{2}\frac{\partial^2 f_j(\cdot)}{\partial \varepsilon^2}\varepsilon^2 + \\ + \Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\Delta x(\varepsilon) + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial x}\Delta x(\varepsilon)\varepsilon + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\Delta t_s(\varepsilon)\varepsilon + o(\| (\Delta x(\varepsilon), \Delta t_s(\varepsilon), \varepsilon) \|^2) \end{aligned} \tag{25}$$

and

$$\begin{aligned} f(x(\varepsilon), t_j^0 + \varepsilon\tau_j, \varepsilon) = f_j(\cdot) + \frac{\partial f_j^\top(\cdot)}{\partial x}\Delta x(\varepsilon) + \frac{\partial f_j^\top(\cdot)}{\partial t}\tau_j\varepsilon + \frac{\partial f_j(\cdot)}{\partial \varepsilon}\varepsilon + \\ + \tfrac{1}{2}\Delta x^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial x^2}\Delta x(\varepsilon) + \tfrac{1}{2}\tau_j^\top\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_j\varepsilon^2 + \tfrac{1}{2}\frac{\partial^2 f_j(\cdot)}{\partial \varepsilon^2}\varepsilon^2 + \\ + \tau_j^\top\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\Delta x(\varepsilon)\varepsilon + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial x}\Delta x(\varepsilon)\varepsilon + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_j\varepsilon^2 + o(\| (\Delta x(\varepsilon), \varepsilon) \|^2). \end{aligned} \tag{26}$$

Note that, due to Assumption 2.3, in the above expansions, we have $o(\| (\Delta x(\varepsilon), \Delta t_s(\varepsilon), \varepsilon) \|^2) = o(\varepsilon^2)$ and $o(\| (\Delta x(\varepsilon), \varepsilon) \|^2) = o(\varepsilon^2)$.

Since $\tau_j \in K(j)$, it holds that $\frac{\partial f_j^\top(\cdot)}{\partial t}\tau_j = 0$. Multiplying both sides of (7) by $\Delta t_s(\varepsilon)$ and using (18), we get $\frac{\partial f_j^\top(\cdot)}{\partial t}\Delta t_s(\varepsilon) \leqslant 0$. Consequently, it follows from (24)–(26) that

$$\begin{aligned} \tfrac{1}{2}\Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t^2}\Delta t_s(\varepsilon) + \Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\Delta x(\varepsilon) + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\Delta t_s(\varepsilon)\varepsilon + o(\varepsilon^2) \geqslant \\ \geqslant \tfrac{1}{2}\tau_j^\top\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_j\varepsilon^2 + \tau_j^\top\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\Delta x(\varepsilon)\varepsilon + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_j\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

Dividing both sides of this inequality by ε^2 and passing to the limit, we obtain

$$\frac{1}{2}\tau_s^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_s^* + \tau_s^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\varkappa^* + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_s^* \geqslant \frac{1}{2}\tau_j^\top\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_j + \tau_j^\top\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\varkappa^* + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_j.$$

But this contradicts inequality (23). Hence, we have proved that for any $j \in \bar{J}$, each vector τ_s^*, $s \in S_j$, is a global optimal solution to problem (17). □

It was proved above that $y^0 \in Y^0$ and $\varkappa^* \in \mathcal{X}^*$. Taking these inclusions into account along with the fact that, by construction, $y_j^0 = 0$ for $j \in J^0 \setminus \bar{J}$ and that $\varkappa^*$ satisfies (22), we conclude that

$$\begin{gathered} y_j^0 = 0 \quad \forall j \in J^0 \setminus (J_*^* \cap \bar{J}), \\ q_j^\top \varkappa^* + \omega_j = 0,\ j \in \bar{J}, \quad q_j^\top \varkappa^* + \omega_j = 0,\ j \in J_*^* \setminus \bar{J}, \quad q_j^\top \varkappa^* + \omega_j \leqslant 0,\ j \in J_0^* \setminus \bar{J}. \end{gathered} \tag{27}$$

To derive further properties of the vector $y^0 = (y_j^0,\ j \in J^0)$ defined in (5), we introduce one more assumption.

Assumption 3.2. *Suppose the following condition holds*

$$\left(\varkappa^{*\top} A_j + c_j^\top\right) l \leqslant 0, \quad \forall l \in \Delta K(j) := \{l \in K(j) \setminus \{0\} : l^\top D_j l = 0\}, \quad \forall j \in J_*^* \setminus \bar{J}.$$

Note that for $j \in J_*^* \setminus \bar{J}$, this condition is considered to be satisfied if $\Delta K(j) = \varnothing$. Thus, Assumption 3.2 holds whenever the inequalities (8) are strict.

According to [25], this assumption is equivalent to the following condition: for each $j \in J_*^* \setminus \bar{J}$, problem (17) admits an optimal solution.

Theorem 3.3. *Let Assumptions* 2.1–2.3, 3.2 *be fulfilled. Then the vector* $y^0 = (y^0_j,\ j \in J^0)$ *defined in* (5) *is an optimal solution to the following problem*

$$\max_{y \in Y^0} \Big(\sum_{j \in J^*_*} y_j \Omega_j(\varkappa^*, \rho_j(\varkappa^*)) + \Omega_0(\varkappa^*) \Big). \tag{28}$$

Proof. Problem (28) can be rewritten in the form

$$(\mathrm{LP}^*): \quad \max \sum_{j \in J^*_*} y_j W_j(\varkappa^*), \quad \text{s. t.} \quad \sum_{j \in J^*_*} q_j y_j + \frac{\partial f_*(x^0, 0)}{\partial x} = 0, \quad y_j \geqslant 0,\ j \in J^*_*;$$

where $W_j(\varkappa^*) := \Omega_j(\varkappa^*, \rho_j(\varkappa^*))$, $j \in J^*_*$, and the functions $\rho_j(\varkappa)$, $\Omega_j(\varkappa, \rho_j)$ are defined in (16) and (15).

Let us write the dual problem to problem (LP^*):

$$(\mathrm{DLP}^*): \quad \min \frac{\partial f_*(x^0, 0)^\top}{\partial x} w, \quad \text{s. t.} \quad q_j^\top w + W_j(\varkappa^*) \leqslant 0,\ j \in J^*_*.$$

The feasible set of problem (LP^*) coincides with that of problem (LP). Therefore, it is nonempty and bounded. By standard results from linear programming theory, the feasible set of dual problem (DLP^*) is also nonempty.

First, let us show that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \frac{\partial f_j^\top(\cdot)}{\partial t} \Delta t_s(\varepsilon) = 0, \quad j \in J^*_* \cap \bar{J}. \tag{29}$$

Consider $j \in J^*_* \cap \bar{J}$. Since $\Delta t_s(\varepsilon) := t_s(\varepsilon) - t^0_j = \varepsilon(\tau^*_s + \Delta\tau_s(\varepsilon))$, where $\|\Delta\tau_s(\varepsilon)\| \to 0$ as $\varepsilon \to 0$, and $\frac{\partial f_j^\top(\cdot)}{\partial t} \tau^*_s = 0$ for $s \in S_j$, $j \in \bar{J}$ (see (19)), we need to show that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\partial f_j^\top(\cdot)}{\partial t} \Delta\tau_s(\varepsilon) = 0. \tag{30}$$

The direction $\Delta t_s(\varepsilon)$ is feasible at the point t^0_j in the set T, and t^0_j solves the problem $\max_{t \in T} f(x^0, t, 0)$. Hence, by the first-order optimality conditions, we have the following

$$\frac{\partial f_j^\top(\cdot)}{\partial t} \Delta t_s(\varepsilon) \leqslant 0 \quad \Longrightarrow \quad \frac{\partial f_j^\top(\cdot)}{\partial t} \Delta\tau_s(\varepsilon) \leqslant 0. \tag{31}$$

The direction $-\Delta t_s(\varepsilon)$ is feasible at the point $t_s(\varepsilon)$ in T, and $t_s(\varepsilon)$ solves the problem $\max_{t \in T} f(x(\varepsilon), t, \varepsilon)$. Therefore, by the first-order optimality conditions for this problem, we have

$$\frac{\partial f^\top(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial t} \Delta t_s(\varepsilon) \geqslant 0.$$

Hence,

$$0 \leqslant \left(\frac{\partial f_j(\cdot)}{\partial t} + \varepsilon \left(\frac{\partial^2 f_j(\cdot)}{\partial t \partial x} \varkappa^* + \frac{\partial^2 f_j(\cdot)}{\partial t^2} \tau^*_s + \frac{\partial^2 f_j(\cdot)}{\partial t \partial \varepsilon} \right) + o(\varepsilon) \right)^\top (\tau^*_s + \Delta\tau_s(\varepsilon)). \tag{32}$$

It was noted above that $\frac{\partial f_j^\top(\cdot)}{\partial t} \tau^*_s = 0$ and τ^*_s is an optimal solution to problem (17). Then from the first-order optimality conditions for τ^*_s in problem (17), we get

$$\left(\frac{\partial^2 f_j(\cdot)}{\partial t \partial x} \varkappa^* + \frac{\partial^2 f_j(\cdot)}{\partial t^2} \tau^*_s + \frac{\partial^2 f_j(\cdot)}{\partial t \partial \varepsilon} \right)^\top \tau^*_s = 0.$$

Consequently, the inequality (32) can be rewritten as follows

$$0 \leqslant \left(\frac{\partial f_j(\cdot)}{\partial t} + \varepsilon \left(\frac{\partial^2 f_j(\cdot)}{\partial t \partial x} \varkappa^* + \frac{\partial^2 f_j(\cdot)}{\partial t^2} \tau^*_s + \frac{\partial^2 f_j(\cdot)}{\partial t \partial \varepsilon} \right) + o(\varepsilon) \right)^\top \Delta\tau_s(\varepsilon).$$

By dividing both sides of the preceding inequality by ε and passing to the limit as $\varepsilon \to 0$, keeping in mind that $\Delta\tau_s(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain

$$0 \leqslant \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\partial f_j^\top(\cdot)}{\partial t} \Delta\tau_s(\varepsilon).$$

Together with (31), this inequality implies (30) and, therefore, (29).

It follows from Assumption 2.3 that the vector $x(\varepsilon)$ admits the representation

$$x(\varepsilon) = x^0 + \varkappa^*\varepsilon + \varepsilon w(\varepsilon) = x^0 + \varkappa^*\varepsilon + \varepsilon^2\tilde{w}(\varepsilon), \tag{33}$$

where $w(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\tilde{w}(\varepsilon) := \frac{w(\varepsilon)}{\varepsilon}$.

Consider $j \in J_*^* \cap \bar{J}$. Then, for each $s \in S_j$, it follows from (25), where $o(\|(\Delta x(\varepsilon), \Delta t_s(\varepsilon), \varepsilon)\|^2) = o(\varepsilon^2)$, that

$$0 = \varepsilon^2 \frac{\partial f_j^\top(\cdot)}{\partial x}\tilde{w}(\varepsilon) + \frac{\partial f_j^\top(\cdot)}{\partial t}\Delta t_s(\varepsilon) + \frac{1}{2}\varepsilon^2\Big(\varkappa^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial x^2}\varkappa^* + \frac{\partial^2 f_j(\cdot)}{\partial \varepsilon^2} +$$
$$+2\frac{\partial^2 f_j(\cdot)}{\partial \varepsilon \partial x}\varkappa^* + 2\varkappa^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial x \partial t}\tau_s^* + \tau_s^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_s^* + 2\frac{\partial^2 f_j(\cdot)}{\partial \varepsilon \partial t}\tau_s^*\Big) + o(\varepsilon^2) \implies$$
$$\implies 0 = f(x(\varepsilon), t_s(\varepsilon), \varepsilon) = \varepsilon^2 \frac{\partial f_j^\top(\cdot)}{\partial x}\tilde{w}(\varepsilon) + \frac{\partial f_j^\top(\cdot)}{\partial t}\Delta t_s(\varepsilon) + \varepsilon^2 W_j(\varkappa^*) + o(\varepsilon^2). \tag{34}$$

Here, we used the relation $\Delta x(\varepsilon) := x(\varepsilon) - x^0 = \varkappa^*\varepsilon + \varepsilon w(\varepsilon)$ (see (33)), the equality $\frac{\partial f_j^\top(\cdot)}{\partial x}\varkappa^* = 0$, and the fact that the vector τ_s^* solves problem (17).

Now consider $j \in J_*^* \setminus \bar{J}$, and let $\tau_j^* \in K(j)$ be an optimal solution to the corresponding problem (17). By Assumption 3.2, such a solution exists. Hence,

$$\frac{\partial f_j^\top(\cdot)}{\partial t}\tau_j^* = 0, \quad \rho_j(\varkappa^*) = \varkappa^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial x \partial t}\tau_j^* + \frac{1}{2}\tau_j^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_j^* + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_j^*. \tag{35}$$

Since $j \notin \bar{J}$, it follows from the assumption that, for all sufficiently small $\varepsilon > 0$, we have $t_j^0 + \varepsilon\tau_j^* \in T$ and $f(x(\varepsilon), t_j^0 + \varepsilon\tau_j^*, \varepsilon) < 0$. Thus,

$$f(x(\varepsilon), t_j^0 + \varepsilon\tau_j^*, \varepsilon) = \varepsilon^2 \frac{\partial f_j^\top(\cdot)}{\partial x}\tilde{w}(\varepsilon) + \varepsilon\frac{\partial f_j^\top(\cdot)}{\partial t}\tau_j^* + \frac{1}{2}\varepsilon^2\Big(\varkappa^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial x^2}\varkappa^* + \frac{\partial^2 f_j(\cdot)}{\partial \varepsilon^2} + 2\frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial x}\varkappa^* +$$
$$+2\varkappa^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial x \partial t}\tau_j^* + \tau_j^{*\top}\frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_j^* + 2\frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_j^*\Big) + o(\varepsilon^2) \implies$$
$$\implies f(x(\varepsilon), t_j^0 + \varepsilon\tau_j^*, \varepsilon) = \varepsilon^2 \frac{\partial f_j^\top(\cdot)}{\partial x}\tilde{w}(\varepsilon) + \varepsilon^2 W_j(\varkappa^*) + o(\varepsilon^2) < 0. \tag{36}$$

Here, we used relations (35), (33), and (27). Taking into account (29), we divide both sides of relations (34) and (36) by ε^2. This yields

$$q_j^\top\tilde{w}(\varepsilon) + W_j(\varkappa^*) = \tilde{O}(\varepsilon),\ j \in J_*^* \cap \bar{J}; \quad q_j^\top\tilde{w}(\varepsilon) + W_j(\varkappa^*) = d_j(\varepsilon) + \tilde{O}(\varepsilon),\ d_j(\varepsilon) \leqslant 0,\ j \in J_*^* \setminus \bar{J}, \tag{37}$$

where $\tilde{O}(\varepsilon)$ denotes a function satisfying $\tilde{O}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Consider the convex quadratic programming problem

$$(\mathrm{QP}_*): \quad \min_w (w - \tilde{w}(\varepsilon))^\top (w - \tilde{w}(\varepsilon)), \quad \text{s. t.} \quad q_j^\top w + W_j(\varkappa^*) \leqslant 0,\ j \in J_*^*.$$

As observed above, the feasible set of this problem is nonempty. Therefore, this problem admits a unique optimal solution, denoted by $w^0(\varepsilon)$. The optimality conditions for $w^0(\varepsilon)$ in (QP_*) imply the existence of a vector $\lambda(\varepsilon) = (\lambda_j(\varepsilon),\ j \in J_*^*)^\top$ such that

$$w^0(\varepsilon) + Q\lambda(\varepsilon) - \tilde{w}(\varepsilon) = 0, \quad \lambda(\varepsilon) \geqslant 0, \quad \lambda^\top(\varepsilon)(Q^\top w^0(\varepsilon) + W(\varkappa^*)) = 0,$$

where $Q = (q_j,\ j \in J_*^*)$ and $W(\varkappa^*) = (W_j(\varkappa^*),\ j \in J_*^*)^\top$. Hence,

$$\tilde{w}(\varepsilon) = w^0(\varepsilon) + Q\lambda(\varepsilon) \quad \implies \quad Q^\top\tilde{w}(\varepsilon) = Q^\top w^0(\varepsilon) + Q^\top Q\lambda(\varepsilon) \implies \tag{38}$$
$$\implies Q^\top\tilde{w}(\varepsilon) + W(\varkappa^*) = Q^\top w^0(\varepsilon) + W(\varkappa^*) + Q^\top Q\lambda(\varepsilon) \implies$$
$$\implies \lambda^\top(\varepsilon)\Big(Q^\top\tilde{w}(\varepsilon) + W(\varkappa^*)\Big) = \lambda^\top(\varepsilon)Q^\top Q\lambda(\varepsilon). \tag{39}$$

Let us show that

$$\lim_{\varepsilon \to 0} \|\lambda(\varepsilon)\| = 0. \tag{40}$$

If $\lambda(\varepsilon) = 0$ for all sufficiently small $\varepsilon > 0$, then equality (40) is evident. Suppose that $\lambda(\varepsilon) \neq 0$ for all sufficiently small $\varepsilon > 0$, and set $\lambda^* := \lim_{\varepsilon\to 0} \frac{\lambda(\varepsilon)}{\|\lambda(\varepsilon)\|}$. Assume that $\lim_{\varepsilon\to 0} \|\lambda(\varepsilon)\| > 0$. Dividing both sides of equality (39) by $\|\lambda(\varepsilon)\|^2$ and passing to the limit while taking into account relations (37), we obtain

$$\sum_{j\in J_*^*\setminus\bar{J}} \lambda_j^* \bar{d}_j = \lambda^{*\top} Q^\top Q\lambda^* \geqslant 0, \quad \text{where } \bar{d}_j = \lim_{\varepsilon\to 0} d_j(\varepsilon)/\|\lambda(\varepsilon)\| \leqslant 0,\ j \in J_*^* \setminus \bar{J}.$$

The above relations imply $Q\lambda^* = 0$, $\lambda^* \geqslant 0$, and $\lambda^* \neq 0$. However, this contradicts the fact that the feasible set of problem (LP_*) is bounded (see problem *(LP)* and its properties). Hence, equality (40) holds.

Relations (38) and equality (40) imply that

$$\tilde{w}(\varepsilon) = w^0(\varepsilon) + \tilde{O}(\varepsilon). \tag{41}$$

By construction, $\sum_{j\in J_*^*} \bar{y}_j q_j = \sum_{j\in J_*^*\cap\bar{J}} y_j^0 q_j$, where the vector $\bar{y} = (\bar{y}_j,\ j \in J^0)^\top \in Y^0$ and satisfies (12). It follows from this equality and relations (37) that

$$\sum_{j\in J_*^*} \bar{y}_j q_j^\top \tilde{w}(\varepsilon) = \sum_{j\in J_*^*\cap\bar{J}} y_j^0 q_j^\top \tilde{w}(\varepsilon) = -\sum_{j\in J_*^*\cap\bar{J}} y_j^0 W_j(\varkappa^*) + \tilde{O}(\varepsilon).$$

Hence,

$$\sum_{j\in J_*^*} \bar{y}_j q_j^\top \tilde{w}(\varepsilon) = \bar{y}_*^\top Q^\top \tilde{w}(\varepsilon) = \mathrm{const} + \tilde{O}(\varepsilon), \tag{42}$$

where $\mathrm{const} := -\sum_{j\in J_*^*\cap\bar{J}} y_j^0 W_j(\varkappa^*)$ and $\bar{y}_* = (\bar{y}_j,\ j \in J_*^*)^\top$.

The feasible set $\mathcal{W} := \{w \in \mathbb{R}^n : Q^\top w + W(\varkappa^*) \leqslant 0\}$ of problems (DLP_*) and (QP_*) is a polyhedron. Hence, it admits a V-representation; that is, there exist finite sets of n-vectors $\mathbf{b}_i$, $i \in I_*$, and $\mathbf{a}_i$, $i \in I_0 \cup I_1$, such that

$$q_j^\top \mathbf{a}_i = 0,\ j \in J_*^*,\ i \in I_0; \quad q_j^\top \mathbf{a}_i \leqslant 0,\ j \in J_*^*,\ \sum_{j\in J_*^*} q_j^\top \mathbf{a}_i = -1,\ i \in I_1; \tag{43}$$

$$\mathcal{W} = \{w \in \mathbb{R}^n : w = \mathcal{B}\gamma + \mathcal{A}_0\alpha + \mathcal{A}_1\beta,\ \gamma \in \Gamma,\ \beta \geqslant 0\},$$

$$\text{where } \mathcal{B} = (\mathbf{b}_i,\ i \in I_*),\ \gamma = (\gamma_i,\ i \in I_*),\ \Gamma = \{\gamma : \gamma_i \geqslant 0,\ i \in I_*,\ \sum_{i\in I_*} \gamma_i = 1\},$$

$$\mathcal{A}_0 = (\mathbf{a}_i,\ i \in I_0),\ \alpha = (\alpha_i,\ i \in I_0),\ \mathcal{A}_1 = (\mathbf{a}_i,\ i \in I_1),\ \beta = (\beta_i,\ i \in I_1).$$

Since $w^0(\varepsilon) \in \mathcal{W}$, there exist $\gamma(\varepsilon) \in \Gamma$, $\alpha(\varepsilon)$ and $\beta(\varepsilon) \geqslant 0$ such that

$$w^0(\varepsilon) = \mathcal{B}\gamma(\varepsilon) + \mathcal{A}_0\alpha(\varepsilon) + \mathcal{A}_1\beta(\varepsilon). \tag{44}$$

It follows from the last representation, together with relations (41), (42), and the equality $Q^\top\mathcal{A}_0 = 0$ (see (43)), that

$$\mathrm{const} + \tilde{O}(\varepsilon) = \bar{y}_*^\top Q^\top \tilde{w}(\varepsilon) = \bar{y}_*^\top Q^\top w^0(\varepsilon) + \tilde{O}(\varepsilon) = \bar{y}_*^\top Q^\top \mathcal{B}\gamma(\varepsilon) + \bar{y}_*^\top Q^\top \mathcal{A}_1\beta(\varepsilon) + \tilde{O}(\varepsilon).$$

Consequently,

$$\bar{y}_*^\top Q^\top \mathcal{B}\gamma(\varepsilon) + \bar{y}_*^\top Q^\top \mathcal{A}_1\beta(\varepsilon) = \mathrm{const} + \tilde{O}(\varepsilon). \tag{45}$$

We now examine the term $\bar{y}_*^\top Q^\top \mathcal{A}_1\beta(\varepsilon)$

$$\bar{y}_*^\top Q^\top \mathcal{A}_1\beta(\varepsilon) = \sum_{j\in J_*^*} \bar{y}_j q_j^\top \sum_{i\in I_1} \mathbf{a}_i\beta_i(\varepsilon) = \sum_{i\in I_1} \beta_i(\varepsilon) \sum_{j\in J_*^*} \bar{y}_j q_j^\top \mathbf{a}_i = \sum_{i\in I_1} \beta_i(\varepsilon)\sigma_i, \tag{46}$$

where, with respect to relations (43), $-\bar{y}_{\max} \leqslant \sigma_i := \sum_{j\in J_*^*} \bar{y}_j q_j^\top \mathbf{a}_i \leqslant -\bar{y}_{\min}$, $\bar{y}_{\max} := \max_{j\in J_*^*} \bar{y}_j \geqslant$ $\geqslant \bar{y}_{\min} := \min_{j\in J_*^*} \bar{y}_j > 0$. Since $\gamma(\varepsilon) \in \Gamma$ and the set Γ is bounded, it follows from (45) and (46) that there exists $M > 0$ such that $0 \leqslant \beta_i(\varepsilon) \leqslant M$ for $i \in I_1$. Hence, without loss of generality, we may assume that

$$\gamma(\varepsilon) \to \gamma^* \in \Gamma, \quad \beta_i(\varepsilon) \to \beta_i^*, \quad 0 \leqslant \beta_i^* \leqslant M,\ i \in I_1, \quad \text{as } \varepsilon \to 0.$$

It follows from (41), (44), and (37) that

$$q_j^\top \tilde{w}(\varepsilon) + W_j(\varkappa^*) = q_j^\top [\mathcal{B}\gamma(\varepsilon) + \mathcal{A}_0\alpha(\varepsilon) + \mathcal{A}_1\beta(\varepsilon)] + W_j(\varkappa^*) + \tilde{O}(\varepsilon) =$$

$$= q_j^\top [\mathcal{B}\gamma(\varepsilon) + \mathcal{A}_1\beta(\varepsilon)] + W_j(\varkappa^*) + \tilde{O}(\varepsilon) = \begin{cases} \tilde{O}(\varepsilon), & j \in J_*^* \cap \bar{J}; \\ d_j(\varepsilon) + \tilde{O}(\varepsilon), & j \in J_*^* \setminus \bar{J}. \end{cases}$$

Passing to the limit as $\varepsilon \to 0$ in the last expression and keeping in mind that $d_j(\varepsilon) \leqslant 0$ for $j \in J_*^* \setminus \bar{J}$, we obtain

$$q_j^\top \bar{w}^* + W_j(\varkappa^*) \begin{cases} = 0, & j \in J_*^* \cap \bar{J}; \\ \leqslant 0, & j \in J_*^* \setminus \bar{J}, \end{cases} \tag{47}$$

where $\bar{w}^* = \mathcal{B}\gamma^* + \mathcal{A}_1\beta_*$ and $\beta_* = (\beta_i^*,\ i \in I_1)^\top$.

By construction, the vector $(y_j^0,\ j \in J_*^*)$ is feasible for problem (LP_*) with $y_j^0 \geqslant 0$ for $j \in J_*^* \cap \bar{J}$ and $y_j^0 = 0$ for $j \in J_*^* \setminus \bar{J}$. This fact, combined with (47), implies that the vector $(y_j^0,\ j \in J_*^*)$ is an optimal solution to problem (LP_*) and the vector $\bar{w}^*$ is an optimal solution to problem (DLP_*). Hence, we have proved that the vector $(y_j^0,\ j \in J^0)$ solves problem (28).

□

3.2. Properties of the primal solution function derivative $\varkappa^*$

The purpose of this subsection is to show that the vector $\varkappa^*$, defined in (9), is an optimal solution to a certain linear-quadratic problem. This problem can be formulated solely in terms of the optimal solution x^0 to the unperturbed problem $(\mathrm{SIP}(0))$, which is a limit point of the sequence $x(\varepsilon)$ as $\varepsilon \to 0$.

Using notations (11) and (15), let us consider the following SIP problem:

$$\begin{aligned} (\mathrm{SIP}^*): \quad & \sum_{j \in J_*^* \cap \bar{J}} y_j^0 \Omega_j(\varkappa, \rho_j) + \Omega_0(\varkappa) \to \min, \\ \text{s. t.} \quad & \frac{1}{2}\tau^\top D_j \tau + c_j^\top \tau + \varkappa^\top A_j \tau - \rho_j \leqslant 0 \quad \forall \tau \in K(j),\ j \in J_*^* \cap \bar{J}; \\ & q_j^\top \varkappa + \omega_j = 0,\ j \in J_*^*;\ \ q_j^\top \varkappa + \omega_j \leqslant 0,\ j \in J_0^*. \end{aligned} \tag{48}$$

In problem (SIP^*), the vector of decision variables is $\phi = (\varkappa \in \mathbb{R}^n,\ \rho_j \in \mathbb{R},\ j \in J_*^* \cap \bar{J})$.

Theorem 3.4. *Let Assumptions* 2.1–2.3 *be fulfilled. Then the vector*

$$\phi^0 = (\varkappa^*,\ \rho_j^* = \rho_j(\varkappa^*),\ j \in J_*^* \cap \bar{J}), \tag{49}$$

where the functions $\rho_j(\varkappa)$, $\varkappa \in \mathbb{R}^n$, $j \in J_*^*$, *are defined in* (16)*, is an optimal solution to problem* (SIP^*).

Proof. It follows from the statements of Theorem 3.1 that the vector $\varkappa^*$ is a feasible solution of (SIP^*).

Consider the following convex quadratic programming problem

$$\begin{aligned} (\mathrm{QP}): \quad & \sum_{j \in J_*^* \cap \bar{J}} y_j^0 \Omega_j(\varkappa, \rho_j) + \Omega_0(\varkappa) \to \min_{\varkappa, \rho}, \\ \text{s. t.} \quad & \frac{1}{2}\tau_s^{*\top} D_j \tau_s^* + c_j^\top \tau_s^* + \varkappa^\top A_j \tau_s^* - \rho_j \leqslant 0, \quad s \in S_j,\ j \in J_*^* \cap \bar{J}; \\ & q_j^\top \varkappa + \omega_j = 0, \quad j \in J_*^*;\ \ q_j^\top \varkappa + \omega_j \leqslant 0, \quad j \in J_0^*. \end{aligned}$$

Let us show that the vector (49) is an optimal solution to problem (QP).

Using notation (11) and setting $\Delta y_s(\varepsilon) := y_s(\varepsilon) - y_s^*$, $s \in S_j$, $j \in \bar{J}$, we obtain from (2)–(6) the following implication

$$\begin{aligned} & 0 = \frac{\partial f_*(x(\varepsilon), \varepsilon)}{\partial x} + \sum_{s \in S(\varepsilon)} y_s^* \frac{\partial f(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial x} + \sum_{s \in S(\varepsilon)} \Delta y_s(\varepsilon) \frac{\partial f(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial x} \implies \\ & \implies 0 = D\Delta x(\varepsilon) + \sum_{j \in \bar{J}} \sum_{s \in S_j} y_s^* A_j \Delta t_s(\varepsilon) + \sum_{j \in \bar{J}} \sum_{s \in S_j} \Delta y_s(\varepsilon) q_j + c\varepsilon + o(\| (\Delta x(\varepsilon), \Delta t_s(\varepsilon), s \in S_j, j \in \bar{J}, \varepsilon) \|). \end{aligned}$$

Taking into account Assumption 2.3, dividing both sides of the last equality by ε, and passing to the limit, we obtain

$$D\varkappa^* + \sum_{j \in \bar{J}} \sum_{s \in S_j} A_j y_s^* \tau_s^* + \lim_{\varepsilon \to 0} \sum_{j \in \bar{J}} q_j \sum_{s \in S_j} \frac{\Delta y_s(\varepsilon)}{\varepsilon} + c = 0. \tag{50}$$

Denote $c^* = D\varkappa^* + \sum_{j\in\bar J}\sum_{s\in S_j} A_j y_s^* \tau_s^* + c$, $Q^* = (q_j,\ j\in\bar J)$,

$$\Delta y_j(\varepsilon) = \sum_{s\in S_j} \Delta y_s(\varepsilon),\ j\in\bar J,\ \Delta y(\varepsilon) = (\Delta y_j(\varepsilon),\ j\in\bar J),\ z(\varepsilon) = Q^*\frac{\Delta y(\varepsilon)}{\varepsilon} + c^*.$$

Then it follows from (50) that $z(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Notice that $y_s^* = 0$ for $s\in S_j$ and $j\in\bar J\setminus J_*^*$. Hence, $\Delta y_j(\varepsilon)\geqslant 0$ for all $j\in\bar J\setminus J_*^*$. Applying Proposition 6.1 (see Appendix), we conclude that there exist numbers η_j^0, $j\in\bar J$, such that $Q^*(\eta_j^0,\ j\in\in\bar J)^\top + c^* = 0$ and $\eta_j^0\geqslant 0$ for $j\in\bar J\setminus J_*^*$.

By setting $\eta_j^0 = 0$ for all $j\in J^0\setminus\bar J$, we conclude that the following relations hold true:

$$D\varkappa^* + \sum_{j\in J_*^*\cap\bar J}\sum_{s\in S_j} A_j y_s^*\tau_s^* + \sum_{j\in J^0} q_j\eta_j^0 + c = 0, \tag{51}$$
$$y_s^*\geqslant 0,\ s\in S_j,\quad \sum_{s\in S_j} y_s^* = y_j^0,\quad j\in J_*^*\cap\bar J,\quad \eta_j^0\geqslant 0,\ j\in J_0^*.$$

It follows from Theorem 3.1 that $\varkappa^*\in\mathcal{X}^*$ (see (13)) and the following equalities take place:

$$\frac{1}{2}\tau_s^{*\top} D_j\tau_s^* + c_j^\top\tau_s^* + \varkappa^{*\top} A_j\tau_s^* - \rho_j^* = 0,\quad s\in S_j,\ j\in J_*^*\cap\bar J. \tag{52}$$

Hence, the vector ϕ^0 is a feasible solution to problem (QP).

It follows from (51) and (52) that the vector ϕ^0 satisfies the necessary and sufficient optimality conditions for the convex problem (QP). Therefore, ϕ^0 solves problem (QP).

Since $\tau_s^*\in K(j)$ for $s\in S_j$, $j\in J_*^*\cap\bar J$, the feasible set of problem (SIP*) is contained in the feasible set of problem (QP). Thus, taking into account the feasibility of the vector ϕ^0 for problem (SIP*) and its optimality in problem (QP), we conclude that ϕ^0 is an optimal solution to problem (SIP*) as well. □

Corollary 3.5. *Let Assumptions* 2.1–2.3 *be fulfilled. Suppose that the following limits exist:*

$$\lim_{\varepsilon\to 0}\frac{y_s(\varepsilon) - y_s^*}{\varepsilon} =: \Delta\eta_s,\quad s\in S_j,\ j\in\bar J.$$

Then these limits, together with those defined in (4)*,* (9)*, and* (10) *solve the following system:*

$$D\varkappa^* + \sum_{j\in\bar J}\sum_{s\in S_j} A_j y_s^*\tau_s^* + \sum_{j\in\bar J} q_j\sum_{s\in S_j}\Delta\eta_s + c = 0,$$
$$y_s^*\geqslant 0,\ s\in S_j,\quad \sum_{s\in S_j} y_s^* = y_j^0,\quad j\in\bar J,\quad \Delta\eta_s\geqslant 0,\ s\in S_j,\ j\in\{i\in\bar J : y_j^0 = 0\},$$
$$q_j^\top\varkappa^* + \omega_j = 0,\quad j\in\bar J\cap J_*^*,\quad q_j^\top\varkappa^* + \omega_j\leqslant 0,\quad j\in J_0^*\setminus\bar J.$$

Consider the minimax problem:

$$F(\varkappa) := \Big[\Omega_0(\varkappa) + \max_{y\in Y^0}\sum_{j\in J^0} y_j\left(\bar\Omega_j(\varkappa) + \max_{\tau\in K(j)}\Psi_j(\tau,\varkappa)\right)\Big] \to \min_{\varkappa\in\mathcal{X}^*}, \tag{53}$$

where $\bar\Omega_j(\varkappa) = \Omega_j(\varkappa,\rho) - \rho$, $j\in J^0$, the functions $\Omega_j(\varkappa,\rho)$, $j\in J^0$, and $\Omega_0(\varkappa)$ are defined in (15), the functions $\Psi_j(\tau,x)$, $j\in J^0$, are defined in (14), Y^0 is the set of optimal solutions to problem (LP), and $\mathcal{X}^*$ is the set of optimal solutions to problem (DLP).

Notice that, to formulate this problem, one needs to know only the optimal solution x^0 to the unperturbed problem (SIP(0)), which is the limit point of the sequence $x(\varepsilon)$ as $\varepsilon\to 0$.

Theorem 3.6. *Let Assumptions* 2.1–2.3 *and* 3.2 *be fulfilled. Then the vector $\varkappa^*$ is an optimal solution to problem* (53)*.*

Proof. Note that it follows from Assumptions 2.1–2.3 and Theorems 3.1, 3.4 that $\varkappa^*\in\mathcal{X}^*$ and $F(\varkappa^*) < +\infty$. Hence, without loss of generality, we can restrict our consideration in problem (53) to

the feasible solutions satisfying the condition $F(x) < +\infty$. Under this condition, problem (53) can be rewritten in the following form:

$$
\begin{aligned}
(\mathrm{SIP}^{**}): \qquad & \beta \to \min, \\
\text{s. t.} \quad & \sum_{j\in J_*^*} y_j \Omega_j(x,\rho_j) + \Omega_0(x) - \beta \leqslant 0 \quad \forall y \in Y^0; \\
& \Psi_j(\tau,x) - \rho_j \leqslant 0 \quad \forall \tau \in K(j),\ j \in J_*^*; \\
& q_j^\top x + \omega_j = 0, \quad j \in J_*^*;\ q_j^\top x + \omega_j \leqslant 0, \quad j \in J_0^*.
\end{aligned}
$$

In this problem, the vector of decision variables is

$$
(x \in \mathbb{R}^n,\ \rho_j \in \mathbb{R},\ j \in J_*^*,\ \beta \in \mathbb{R}). \tag{54}
$$

As before, let us define the functions

$$
\rho_j(x) := \max_{\tau\in K(j)} \Psi_j(\tau,x), \quad j \in J_*^*, \quad \beta(x) := \max_{y\in Y^0} \Big(\sum_{j\in J_*^*} y_j \Omega_j(x,\rho_j(x)) + \Omega_0(x) \Big).
$$

Notice that if vector (54) is a feasible solution of problem (SIP^{**}), then $\rho_j(x) < +\infty$ for all $j \in J_*^*$.

Let x be a feasible solution of problem (53). It is easy to verify that the vector $(x,\ \rho_j(x),\ j \in J_*^*,\ \beta(x))$ is a feasible solution of (SIP^{**}), and for any feasible solution $(x,\ \rho_j,\ j \in J_*^*,\ \beta)$ of (SIP^{**}), we have $\beta(x) \leqslant \beta$. Moreover, we have $F(x) = \beta(x)$.

Let $\tilde{x} \in \mathcal{X}^*$ be a feasible solution of problem (53) such that $F(\tilde{x}) < +\infty$. Then, taking into account that $y^0 \in Y^0$, we obtain

$$
F(\tilde{x}) = \beta(\tilde{x}) = \max_{y\in Y^0} \Big(\sum_{j\in J_*^*} y_j \Omega_j(\tilde{x},\rho_j(\tilde{x})) + \Omega_0(\tilde{x}) \Big) \geqslant \sum_{j\in J_*^*} y_j^0 \Omega_j(\tilde{x},\rho_j(\tilde{x})) + \Omega_0(\tilde{x}).
$$

Since Assumption 3.2 is satisfied, problem (17) has an optimal solution for each $j \in J_*^* \setminus \bar{J}$. Hence, without loss of generality, the constraint (48) in problem (SIP^*) can be replaced by the following one:

$$
\frac{1}{2}\tau^\top D_j \tau + c_j^\top \tau + x^\top A_j \tau - \rho_j \leqslant 0 \quad \forall \tau \in K(j),\ j \in J_*^*,
$$

and the vector of decision variables $\phi = (x \in \mathbb{R}^n,\ \rho_j \in \mathbb{R},\ j \in J_*^* \cap \bar{J})$ can be replaced by $(x \in \mathbb{R}^n,\ \rho_j \in \mathbb{R},\ j \in J_*^*)$.

Taking into account this observation, together with the fact that $y_j^0 = 0$ for all $j \in J_*^* \setminus \bar{J}$ and applying Theorem 3.4, we obtain

$$
\sum_{j\in J_*^*} y_j^0 \Omega_j(\tilde{x},\rho_j(\tilde{x})) + \Omega_0(\tilde{x}) \geqslant \sum_{j\in J_*^*} y_j^0 \Omega_j(x^*,\rho_j(x^*)) + \Omega_0(x^*).
$$

Furthermore, by Theorem 3.3, we know that

$$
\sum_{j\in J_*^*} y_j^0 \Omega_j(x^*,\rho_j(x^*)) + \Omega_0(x^*) = \max_{y\in Y^0} \Big(\sum_{j\in J_*^*} y_j \Omega_j(x^*,\rho_j(x^*)) + \Omega_0(x^*) \Big) =: \beta(x^*) = F(x^*).
$$

Hence, we have shown that $F(\tilde{x}) \geqslant F(x^*)$ for any $\tilde{x} \in \mathcal{X}^*$. Since $x^* \in \mathcal{X}^*$, we conclude that x^* is an optimal solution to problem (53). □

4. Strengthening Assumption 3.2

In this section, we consider some assumptions that guarantee that Assumption 3.2 holds.

Assumption 4.1. *There exists a finite limit*

$$
\lim_{\varepsilon\to 0} \frac{x(\varepsilon) - x^0 - \varepsilon x^*}{\varepsilon^2} =: w^*.
$$

Lemma 4.2. *Let Assumptions* 2.1–2.3 *and* 4.1 *be fulfilled. Then Assumption* 3.2 *is fulfilled as well; i. e., for $j \in J_*^* \setminus \bar{J}$, problem* (17) *has an optimal solution.*

Proof. Suppose the contrary: for some $j_0 \in J_*^* \setminus \bar{J}$, the corresponding problem (17) has no optimal solution. Then it follows from [25] that there exists a vector $\bar{\tau} \in \mathbb{R}^p$ such that

$$\bar{\tau} \in K(j_0), \quad \bar{\tau}^\top D_{j_0} \bar{\tau} = 0, \quad (c_{j_0} + A_{j_0}^\top \varkappa^*)\bar{\tau} > 0. \tag{55}$$

Notice that it was proved above that, under Assumption 2.3, relations (27) hold. It follows from these relations that $q_j^\top \varkappa^* + \omega_j = 0$ for $j \in J_*^*$ and hence,

$$q_{j_0}^\top \Delta x(\varepsilon) + \omega_{j_0}\varepsilon = q_{j_0}^\top w^* \varepsilon^2 + o(\varepsilon^2). \tag{56}$$

Since $\bar{\tau} \in K(j_0)$, there exists a number $\varepsilon_* > 0$ such that $t_{j_0}^0 + \varepsilon\bar{\tau} \in T$ for all $0 \leqslant \varepsilon \leqslant \varepsilon_*$. By construction, $f(x^0, t_{j_0}^0, 0) = 0, \quad f(x(\varepsilon), t_{j_0}^0 + \varepsilon\bar{\tau}, \varepsilon) \leqslant 0$, and therefore

$$f(x(\varepsilon), t_{j_0}^0 + \varepsilon\bar{\tau}, \varepsilon) = \tag{57}$$
$$= \varepsilon^2 \big(\tfrac{\partial f_{j_0}^\top(\cdot)}{\partial x} w^* + \tfrac{1}{2} \varkappa^{*\top} \tfrac{\partial^2 f_{j_0}(\cdot)}{\partial x^2} \varkappa^* + \tfrac{1}{2} \tfrac{\partial^2 f_{j_0}(\cdot)}{\partial \varepsilon^2} + \tfrac{\partial^2 f_{j_0}^\top(\cdot)}{\partial \varepsilon \partial x} \varkappa^* + \bar{\tau}^\top \tfrac{\partial^2 f_{j_0}(\cdot)}{\partial t \partial x} \varkappa^* + \tfrac{\partial^2 f_{j_0}^\top(\cdot)}{\partial \varepsilon \partial t} \bar{\tau} \big) + o(\varepsilon^2) \leqslant 0.$$

Here, we took into account Assumption 2.3, the equalities $\frac{\partial f_{j_0}^\top(\cdot)}{\partial t}\bar{\tau} = 0$, $\bar{\tau}^\top D_{j_0}\bar{\tau} = 0$ (see (55)), and (56).

Dividing both sides of inequality (57) by ε^2 and passing to the limit, we obtain

$$\text{const} + (A_{j_0}^\top \varkappa^* + c_{j_0})^\top \bar{\tau} \leqslant 0, \tag{58}$$

where $\text{const} = \frac{\partial f_{j_0}^\top(\cdot)}{\partial x} w^* + \frac{1}{2} \varkappa^{*\top} \frac{\partial^2 f_{j_0}(\cdot)}{\partial x^2} \varkappa^* + \frac{1}{2} \frac{\partial^2 f_{j_0}(\cdot)}{\partial \varepsilon^2} + \frac{\partial^2 f_{j_0}^\top(\cdot)}{\partial \varepsilon \partial x} \varkappa^*$.

Notice that it follows from (55) that without loss of generality, the vector $\bar{\tau}$ can be scaled such that $(A_{j_0}^\top \varkappa^* + c_{j_0})^\top \bar{\tau} \geqslant a$, where $0 < a < \infty$ is any fixed number. However, this contradicts (58). Hence, we have proved that, for any $j \in J_*^* \setminus \bar{J}$, problem (17) has an optimal solution. □

Let us make the following assumption.

Assumption 4.3. *There exists a limit* (9)*, and for each* $j \in J_*^* \cup \bar{J}$*,*

$$\big(A_j^\top \varkappa^* + c_j\big)^\top l < 0 \quad \forall l \in \Delta K(j) := \{l \in K(j) \setminus \{0\} : l^\top D_j l = 0\}. \tag{59}$$

Note that relations (59) imply the fulfillment of Assumption 3.2 and, consequently, the existence of optimal solutions to problem (17) for all $j \in J_*^* \cup \bar{J}$.

Lemma 4.4. *Suppose that Assumption* 4.3 *is fulfilled. Then there exist limits* (10).

Proof. Consider an index $j \in \bar{J}$. It follows from Assumption 4.3 that the corresponding problem (17) has an optimal solution, say τ_j. Since $\tau_j \in K(j)$, there exists a number $\varepsilon_* > 0$ such that $t_j^0 + \varepsilon\tau_j \in T$ for all $0 \leqslant \varepsilon \leqslant \varepsilon_*$.

Recall that, for $s \in S_j$, inequalities (24) and expansions (25) and (26) hold. Since $\tau_j \in K(j)$, we have $\frac{\partial f_j^\top(\cdot)}{\partial t}\tau_j = 0$. Multiplying both sides of (7) by $\Delta t_s(\varepsilon)$ and taking into account (18), we get $\frac{\partial f_j^\top(\cdot)}{\partial t}\Delta t_s(\varepsilon) \leqslant 0$. Consequently, from (24), (25), and (26) it follows:

$$\frac{1}{2}\Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t^2}\Delta t_s(\varepsilon) + \Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\Delta x(\varepsilon) + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\Delta t_s(\varepsilon)\varepsilon + o(\| (\Delta t_s(\varepsilon), \varepsilon) \|^2) \geqslant$$
$$\geqslant \frac{1}{2}\tau_j^\top \frac{\partial^2 f_j(\cdot)}{\partial t^2}\tau_j \varepsilon^2 + \tau_j^\top \frac{\partial^2 f_j(\cdot)}{\partial t \partial x}\Delta x(\varepsilon)\varepsilon + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t}\tau_j \varepsilon^2 + o(\varepsilon^2). \tag{60}$$

It is evident that $I_s(\varepsilon) = I_a(t_s(\varepsilon)) \subset I_j$ for sufficiently small $\varepsilon > 0$. Hence, we have

$$a_i^\top \Delta t_s(\varepsilon) = 0, \quad i \in I_s(\varepsilon), \quad a_i^\top \Delta t_s(\varepsilon) \leqslant 0, \quad i \in I_j \setminus I_s(\varepsilon). \tag{61}$$

Multiplying both sides of (1) by $\Delta t_s(\varepsilon)$ and taking into account (61), we obtain:

$$\frac{\partial f^\top(x(\varepsilon), t_s(\varepsilon), \varepsilon)}{\partial t}\Delta t_s(\varepsilon) = \sum_{i \in I_s(\varepsilon)} a_i^\top \Delta t_s(\varepsilon)\mu_i(s, \varepsilon) = 0. \tag{62}$$

Conditions (61) and (62) imply that $\Delta t_s(\varepsilon) \in K(j)$. Hence, it follows from (8) that

$$\Delta t_s^\top(\varepsilon)\frac{\partial^2 f_j(\cdot)}{\partial t^2}\Delta t_s(\varepsilon) \leqslant 0. \tag{63}$$

Let us divide both sides of (61) by $\|\Delta t_s(\varepsilon)\|$ and pass to the limit. As a result, we obtain

$$a_i^\top \bar{\tau}_s^* \leqslant 0, \quad i \in I_j,\ s \in S_j,\ j \in \bar{J}, \quad \text{with } \bar{\tau}_s^* = \lim_{\varepsilon \to 0} \frac{\Delta t_s(\varepsilon)}{\|\Delta t_s(\varepsilon)\|}.$$

Next, dividing both sides of equality (62) by $\|\Delta t_s(\varepsilon)\|$ and passing to the limit, we get

$$\frac{\partial f_j^\top(\cdot)}{\partial t} \bar{\tau}_s^* = 0, \quad s \in S_j,\ j \in \bar{J}.$$

Hence, we have shown that $\bar{\tau}_s^* \in K(j)$ for $s \in S_j$, $j \in \bar{J}$.

Suppose that $\lim\limits_{\varepsilon \to 0} \frac{\|\Delta t_s(\varepsilon)\|}{\varepsilon} = \infty$. This implies that $\lim\limits_{\varepsilon \to 0} \frac{\varepsilon}{\|\Delta t_s(\varepsilon)\|} = 0$.

Dividing both sides of inequality (60) by $\|\Delta t_s(\varepsilon)\|^2$ and passing to the limit yields $\frac{1}{2} \bar{\tau}_s^{*\top} \frac{\partial^2 f_j(\cdot)}{\partial t^2} \bar{\tau}_s^* \geqslant 0$. This inequality, together with the inclusion $\bar{\tau}_s^* \in K(j)$ and inequalities (8), implies that

$$\frac{1}{2} \bar{\tau}_s^{*\top} \frac{\partial^2 f_j(\cdot)}{\partial t^2} \bar{\tau}_s^* = 0.$$

Now, let us divide both sides of inequality (60) by $\|\Delta t_s(\varepsilon)\| \varepsilon$ and pass to the limit. We then obtain

$$\frac{1}{2} \lim_{\varepsilon \to 0} \Big(\frac{\|\Delta t_s(\varepsilon)\|}{\varepsilon} \frac{\Delta t_s^\top(\varepsilon)}{\|\Delta t_s(\varepsilon)\|} \frac{\partial^2 f_j(\cdot)}{\partial t^2} \frac{\Delta t_s(\varepsilon)}{\|\Delta t_s(\varepsilon)\|} \Big) +$$
$$+ (\bar{\tau}_s^*)^\top \frac{\partial^2 f_j(\cdot)}{\partial t \partial x} \varkappa^* + \frac{\partial^2 f_j^\top(\cdot)}{\partial \varepsilon \partial t} \bar{\tau}_s^* + \frac{o(\|(\Delta t_s(\varepsilon), \varepsilon)\|^2)}{\|\Delta t_s(\varepsilon)\| \varepsilon} \geqslant 0. \tag{64}$$

It follows from Assumption 4.3, the inclusion $\bar{\tau}_s^* \in K(j)$, and the equality $\frac{1}{2} (\bar{\tau}_s^*)^\top \frac{\partial^2 f_j(\cdot)}{\partial t^2} \bar{\tau}_s^* = 0$ that

$$(\bar{\tau}_s^*)^\top \frac{\partial^2 f_j(\cdot)}{\partial t \partial x} \varkappa^* + \frac{\partial^2 f_j(\cdot)}{\partial \varepsilon \partial t} \bar{\tau}_s^{*\top} < 0.$$

From (63), we conclude that

$$\frac{1}{2} \lim_{\varepsilon \to 0} \Big(\frac{\|\Delta t_s(\varepsilon)\|}{\varepsilon} \frac{\Delta t_s^\top(\varepsilon)}{\|\Delta t_s(\varepsilon)\|} \frac{\partial^2 f_j(\cdot)}{\partial t^2} \frac{\Delta t_s(\varepsilon)}{\|\Delta t_s(\varepsilon)\|} \Big) \leqslant 0.$$

However, these inequalities contradict inequality (64). The resulting contradiction proves that $\lim_{\varepsilon \to 0} \frac{\|\Delta t_s(\varepsilon)\|}{\varepsilon} < \infty$, and consequently, there exists a limit $\lim_{\varepsilon \to 0} \frac{\Delta t_s(\varepsilon)}{\varepsilon} =: \tau_s^*$. □

The following corollary is a direct consequence of Theorems 3.3, 3.6 and Lemmas 4.2, 4.4.

Corollary 4.5. *Suppose that one of the following sets of assumptions is fulfilled:*

• *Assumptions* 2.1–2.3 *and* 4.1,

• *Assumptions* 2.1–2.2 *and* 4.3.

Then the vector $y^0 = (y_j^0,\ j \in J^0)$ *defined in* (5) *is an optimal solution to problem* (28) *and* $\varkappa^*$ *is an optimal solution to problem* (53).

5. Comparison with known results

The results obtained in this paper are most closely related to those presented in [3], which investigates parametric problems of both semi-infinite programming and more general optimization problems in Banach spaces. In [3], the analysis of parametric SIP problems primarily focuses on the behavior of the objective function under small perturbations of the parameter ε. For the asymptotic analysis and structural properties of the optimal solution $x(\varepsilon)$, the authors refer to general results established for abstract parametric optimization problems.

Regarding these general results, the following limitations can be highlighted:

a) These results are formulated in a highly complex and abstract framework.

b) The underlying assumptions are difficult to verify in practice.

c) The assumptions in [3] are quite restrictive, as they are tailored to much broader classes of optimization problems rather than specifically to SIP.

Specifically, Theorem 4.101 in [3] provides a formulation that is closest to SIP settings, since it assumes a finite-dimensional decision variable vector. Under certain conditions, this theorem states that

> "For any $o(\varepsilon)$-optimal solution $\tilde{x}(\varepsilon)$ of SIP(ε), every accumulation point of $\frac{\tilde{x}(\varepsilon)-x^0}{\varepsilon}$ as $\varepsilon \to 0$ is an optimal solution of the min-max problem DQ",

where problem DQ serves as an analogue to our problem (53). One of the key assumptions required for this theorem is the second-order regularity condition (see condition (v) of Theorem 4.101 in [3]). This condition is formulated implicitly and is generally difficult to verify.

To overcome this drawback, sufficient conditions ensuring this second-order regularity are provided in [3]. The sufficient condition closest to the structure of our problem and our assumptions can be stated as follows (using the notation introduced above [3, p. 526]):

Let x^0 be an optimal solution to SIP(0). *Suppose that:*

(a) The set of active indices $T_a(x^0,0) = \{t_j^0,\ j \in J^0\}$ is finite.

(b) The set T is second-order regular at every point of $T_a(x^0,0)$.

(c) The second-order growth condition for the lower-level problem holds, i. e., for every $t_j^0 \in T_a(x^0,0)$, there exist a constant $c > 0$ and a neighborhood N_j of t_j^0 such that

$$-f(x^0,t,0) \geqslant c\|t-t_j^0\|^2 \quad \forall t \in N_j.$$

Then the second-order regularity condition (v) in Theorem 4.101 *from* [3] *is satisfied.*

Let us show that if there exists an index $j_0 \in J^0$ such that $\Delta K(j_0) := \{l \in K(j_0) \setminus \{0\} : l^\top D_j l = 0\} \neq \varnothing$, then condition (c) fails to hold.

Indeed, let $l \in \Delta K(j_0)$ with $\|l\| = 1$. Then $\tau(\theta) := t_{j_0}^0 + \theta l \in T$ and $\tau(\theta) \in N_{j_0}$ for sufficiently small $\theta \geqslant 0$. Computing the Taylor expansion yields:

$$-f(x^0,t_{j_0}^0+\theta l,0) = -\Big(\theta\frac{\partial f_{j_0}^\top(\cdot)}{\partial t}l + \frac{1}{2}\theta^2 l^\top \frac{\partial^2 f_{j_0}(\cdot)}{\partial t^2}l + o(\theta^2)\Big) = o(\theta^2).$$

However, the relations $o(\theta^2) \geqslant c\|\tau(\theta) - t_{j_0}^0\|^2 = c\theta^2$ cannot hold for a strictly positive constant c and sufficiently small $\theta \geqslant 0$.

Note that, for our min-max problem (53), we provide an explicit formulation in the form of a specific convex semi-infinite programming problem (SIP**). This problem has been studied in detail in [26; 27].

In our framework, we do not exclude the case where $\Delta K(j_0) \neq \varnothing$. Consequently, our assumptions are weaker than those in [3], and the results presented in this paper cannot be derived as direct consequences of the theorems in [3].

Furthermore, Theorems 3.1–3.4 characterize the asymptotic properties not only of the limit points $\varkappa^*$ of the sequence $\frac{x(\varepsilon)-x^0}{\varepsilon}$, but also of the limit points τ_s^* of the sequence $\frac{t_s(\varepsilon)-t_j^0}{\varepsilon}$ for $s \in S_j$, $j \in \bar{J}$, as well as the limit points $\Delta\eta_s$ of the sequence $\frac{y_s(\varepsilon)-y_s^*}{\varepsilon}$ for $s \in S_j$, $j \in \bar{J}$ as $\varepsilon \to 0$. Namely, it is shown that for $j \in \bar{J}$, the vectors τ_s^*, $s \in S_j$, are optimal solutions to the problem $\max\limits_{\tau \in K(j)} \Psi_j(\tau, \varkappa^*)$, and the vector $\eta^0 = (\eta_j^0 = \sum_{s\in S_j}\Delta\eta_s,\ j \in \bar{J};\ \eta_j^0 = 0,\ j \in J^0 \setminus \bar{J})^\top$ is the vector of optimal Lagrange multipliers for the problem (QP) corresponding to the constraints:

$$q_j^\top \varkappa + \omega_j = 0, \quad j \in J_*^*; \qquad q_j^\top \varkappa + \omega_j \leqslant 0, \quad j \in J_0^*.$$

The derived properties of the derivatives $\varkappa^*$, η^0, and τ_s^* ($s \in S_j$, $j \in \bar{J}$), along with the explicitly constructed optimization problems they solve, are highly convenient for analyzing the inverse problem. That is, given the solution to the unperturbed problem, it is necessary to describe the behavior of the perturbed system, including predicting the (new) number of active indices under perturbation. These topics will be the focus of our future research.

6. Appendix

Proposition 6.1. *Suppose that*

$$Q^*\frac{\Delta y(\varepsilon)}{\varepsilon} + c^* = z(\varepsilon)\ \forall \varepsilon \in (0, \varepsilon_*], \tag{65}$$
$$z(\varepsilon) \to 0,\ \Delta y(\varepsilon) = (\Delta y_k(\varepsilon), k \in P)^\top \to 0 \ as\ \varepsilon \to 0;\ \Delta y_k(\varepsilon) \geqslant 0, k \in I \subset P.$$

Then there exists a vector $y^* = (y^*_k, k \in P)^\top$ *such that*

$$Q^* y^* + c^* = 0,\ y^*_k \geqslant 0,\ k \in I. \tag{66}$$

Proof. Consider the following linear programming problem:

$$(\mathrm{LP}_0): \qquad \max\ \theta \text{ s. t. } Q^* y + \theta c^* = 0,\ y_k \geqslant 0,\ k \in I.$$

This problem has a feasible solution $(y = 0,\ \theta = 0)$. Suppose that there exists another feasible solution $(\bar{y}, \bar{\theta})$ of this problem such that $\bar{\theta} > 0$. Then the vector $y^* = \bar{y}/\bar{\theta}$ satisfies relations (66) and the proposition is proved.

Suppose now that there is no feasible solution $(\bar{y}, \bar{\theta})$ of problem (LP_0) such that $\bar{\theta} > 0$. Then $(y^0 = 0, \theta^0 = 0)$ is an optimal solution to problem (LP_0). Therefore, by the optimality conditions,

$$\Delta^* := Q^{*\top}\lambda^*,\ \Delta^*_k = 0\ \forall k \in P \setminus I,\ \Delta^*_k \geqslant 0\ \forall k \in I,\ \lambda^{*\top} c^* = 1. \tag{67}$$

Multiplying equality (65) by $\lambda^{*\top}$, we obtain $\lambda^{*\top} Q^* \frac{\Delta y(\varepsilon)}{\varepsilon} + \lambda^{*\top} c^* = \lambda^{*\top} z(\varepsilon)$, and taking into account (67), we have

$$\sum_{k \in I} \Delta^*_k \frac{\Delta y_k(\varepsilon)}{\varepsilon} + 1 = \lambda^{*\top} z(\varepsilon) \text{ where } \Delta^*_k \geqslant 0,\ \Delta y_k(\varepsilon) \geqslant 0\ \forall k \in I. \tag{68}$$

Set $\alpha := \lim\limits_{\varepsilon \to 0} \sum\limits_{k \in I} \Delta^*_k \frac{\Delta y_k(\varepsilon)}{\varepsilon} \geqslant 0$. Since $z(\varepsilon) \to 0$ as $\varepsilon \to 0$, passing to the limit in (68) yields $\alpha + 1 = 0$ which is impossible because $\alpha \geqslant 0$. This contradiction completes the proof of the proposition. □

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References

1. Fiacco A. V. *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*. New York, Academic Press, 1983.

2. Levitin E. S. *Perturbation Theory in Mathematical Programming and its Applications*. Moscow, Nauka, 1992 (in Russian).

3. Bonnans J. F., Shapiro A. *Perturbation Analysis of Optimization Problems*. Springer Science & Business Media, 2013.

4. Hettich R., Kortanek K. Semi-infinite programming: Theory, methods and applications. *SIAM Review*, 1993, vol. 35, iss. 3, pp. 380–429.

5. Stecher J., Kiltz, L., Ruess N., Graichen K. Why engineers should care about semi-infinite programming: Nominal versus tolerance-aware geometry optimization of a proportional electromagnetic actuator. *IFAC-PapersOnLine,* 2025, vol. 59, iss. 17, pp. 113–118.

6. Castillo E., Minguez R., Castillo C. Sensitivity analysis in optimization and reliability problems. *Reliability Engineering & System Safety,* 2008, vol. 93, iss. 12, pp. 1788–1800.

7. Ben-Tal A., Nemirovski A. Selected topics in robust convex optimization. *Mathematical Programming*, 2008, vol. 112, iss. 1, pp. 125–158.

8. Puschke J., Djelassi H., Kleinekorte J., Hannemann-Tamás R., Mitsos A. Robust dynamic optimization of batch processes under parametric uncertainty: Utilizing approaches from semi-infinite programs. *Computers & Chemical Engineering,* 2018, vol. 116, pp. 253–267.

9. Bank B., Guddat J., Klatte D., Kummer B., Tammer K. *Non-Linear Parametric Optimization*. Birkhauser, Basel, 1983.

10. Izmailov A. F. Sensitivity of solutions to systems of optimality conditions under the violation of constraint qualifications. *Computational Mathematics and Mathematical Physics,* 2007, vol. 47, iss. 4, pp. 533–554.

11. Dontchev A. L. Rockafellar R. T. Characterizations of Lipschitzian stability in nonlinear programming. *Fiacco A. V. (Ed.). Mathematical Programming with Data Perturbations*, CRC Press, 2020, pp. 65–82.

12. Dempe S. Directional differentiability of optimal solutions under Slater's condition. *Mathematical Programming,* 1993, vol. 59, iss. 1, pp. 49–69.

13. López M. A., Still G. Semi-infinite programming. *European Journal of Operational Research*, 2007, vol. 180, iss. 2, pp. 491–518.

14. Goberna M. A., López M. A. *Post-Optimal Analysis in Linear Semi-Infinite Optimization*. Springer, Briefs in Optimization, 2014.

15. Reemtsen R., Ruckmann J. J. (Eds.). *Semi-Infinite Programming.* New York, Springer Science & Business Media, 2013. 514 p.

16. Goberna M. A., Terlaky T., Todorov M. I. Sensitivity analysis in linear semi-infinite programming via partitions. *Mathematics of Operations Research,* 2010, vol. 35, iss. 1, pp. 14–26.

17. Rockafellar R. T., Wets R. J.-B. *Variational Analysis*. Springer Science & Business Media, 2009.

18. Cánovas M. J., López M. A., Mordukhovich B. S., Parra J. Variational analysis in semi-infinite and infinite programming, I: Stability of linear inequality systems of feasible solutions. *SIAM Journal on Optimization,* 2010, vol. 20, iss. 3, pp. 1504–1526.

19. Poore A. B. Bifurcation problems for some parametric nonlinear programs in Banach spaces. *SIAM Journal on Control and Optimization,* 1996, vol. 34, iss. 6, pp. 1947–1971.

20. Levitin E. S., Polyak B. T. Constrained minimization methods. *Computational Mathematics and Mathematical Physics*, 1966, vol. 6, iss. 5, pp. 1–50 (in Russian).

21. Polyak B. T. *Introduction to Optimization*. New York, Optimization Software, Inc., Publications Division, 1987.

22. Zheng X. Y., Liu S. T. Levitin–Polyak well-posedness and stability for parametric optimization problems. *Journal of Optimization Theory and Applications*, 2021, vol. 189, iss. 1, pp. 112–134.

23. Mitsos A., Gharbaoui B., Stein O. Recent advances in nonconvex semi-infinite programming: Applications and algorithms. *Journal of Computing and Optimization*, 2021, vol. 5, iss. 2, pp. 105–124.

24. López M. A. A brief tutorial on semi-infinite optimization: Theory, models and new analytical constructions. *Top / Springer Nature*, 2026, vol. 34, iss. 1, pp. 1–25.

25. Eaves B. C. On quadratic programming. *Management Science,* 1971, vol. 17, iss. 11, pp. 698–711.

26. Kostyukova O., Tchemisova T., Kurdina M. Existence theorem and optimality conditions for a class of convex semi-infinite problems with noncompact index sets. *Statistics, Optimization & Information Computing,* 2017, vol. 5, iss. 4, pp. 278–294.

27. Kostyukova O., Tchemisova T., Kurdina M. On optimal properties of special nonlinear and semi-infinite problems arising in parametric optimization. *Statistics, Optimization & Information Computing,* 2017, vol. 5, iss. 2, pp. 99–108.