Национальная академия наук Беларуси
Труды Института математики НАН Беларуси. 2026. Том 34. № 1. С. 85–95

UDC 517.956.35 https://doi.org/10.67268/1812-5093-2026-34-1-85-95
EDN: DCQCNW

CLASSICAL SOLUTION TO THE FIRST MIXED PROBLEM FOR A MILDLY QUASILINEAR WAVE EQUATION: A FIXED-POINT APPROACH

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Received: 05.02.2026 **Revised:** 20.03.2026 **Accepted:** 29.05.2026

Keywords: mildly quasilinear wave equation, mixed problem, classical solution, fixed-point principle, matching conditions.

Abstract. For a one-dimensional mildly quasilinear wave equation given in the first quadrant, we consider a mixed problem in which Cauchy conditions are specified on the spatial semi-axis and a Dirichlet condition is specified on the temporal semi-axis. The nonlinearity contains independent variables, the unknown function, and its derivatives. We construct the solution in implicit analytical form as the solution of some integro-differential equations. We prove the solvability of these integro-differential equations using a generalization of the Banach fixed-point theorem. For the problem in question, we prove the uniqueness of the solution and establish the conditions under which its classical solution exists

КЛАССИЧЕСКОЕ РЕШЕНИЕ ПЕРВОЙ СМЕШАННОЙ ЗАДАЧИ ДЛЯ СЛАБО КВАЗИЛИНЕЙНОГО ВОЛНОВОГО УРАВНЕНИЯ: МЕТОД НЕПОДВИЖНОЙ ТОЧКИ

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Поступила: 05.02.2026 **Исправлена:** 20.03.2026 **Принята:** 29.05.2026

Ключевые слова: слабо квазилинейное волновое уравнение, смешанная задача, классическое решение, принцип неподвижной точки, условия согласования.

Аннотация. Для одномерного слабо квазилинейного волнового уравнения, заданного в первом квадранте, рассматривается смешанная задача, в которой на пространственной полуоси задаются условия Коши, а на временной полуоси задается условие Дирихле. Нелинейность содержит независимые переменные, искомую функцию и ее производные. Решение строится в неявном аналитическом виде как решение некоторых интегро-дифференциальных уравнений. Разрешимость интегро-дифференциальных уравнений доказывается с использованием обобщения теоремы Банаха о неподвижной точке. Для рассматриваемой задачи доказывается единственность решения и устанавливаются условия, при выполнении которых существует ее классическое решение.

1. Introduction

We often use various fixed-point theorems and the method of successive approximations to find solutions to initial and mixed problems for nonlinear equations. For example, the classical Banach fixed-point theorem has been used to find a weak solution to the Cauchy problem for the nonlinear wave equation and the nonlinear parabolic equation with nonlinearities of the form $f(\nabla u, \partial_t u, u)$ and $f(u)$, respectively [1]. Many results have been obtained using topological fixed-point theorems, such as the Schauder fixed-point theorem and the Leray–Schauder fixed-point principle, namely: 1) the existence of a weak solution to a nonlinear elliptic equation [2] (nonlinearity of the form $f(u)$) in the space H_0^1; 2) classical solutions of various mixed problems with a nonlinear boundary condition for the semilinear equation of string oscillation in a half-strip [3–5] (nonlinearities of the form $F[\partial_t u(0,t)]$ and $f(u)$); 3) classical solution of the Cauchy–Darboux problem, the first Darboux problem and the second Darboux problem for the nonlinear wave equation [6–8] (nonlinearity of the form $\lambda|u|^{\alpha}u$); 4) existence of a weak solution of a quasilinear elliptic equation [1] (nonlinearity of the form $f(\nabla u)$) in the space $H^2 \cap H_0^1$; 5) solvability of the Dirichlet problem for nonlinear elliptic equations in the Hölder spaces $C^{2,\alpha}$ [9]. Using the fixed-point theorem adapted to modular metric spaces, it is shown that the Cauchy problem for a one-dimensional nonlinear parabolic equation with nonlinearity of the form $f(t,x,u(t,x),\partial_x u(t,x))$ could be solved [2]. Together with

V. I. Korzyuk, the author of this paper used the Banach fixed-point theorem for locally convex spaces to construct classical and weak solutions of the Cauchy problem for a mildly quasilinear wave equation [10].

Various versions of the fixed point principle are closely related to the method of successive approximations. This method has produced several results in the theory of partial differential equations. For example, it has been used to find: 1) a twice continuously differentiable solution to the Cauchy problem in a cone and a truncated cone for a nonlinear wave equation with nonlinearity of the form $F'(|u|^2)u$ [11]; 2) the classical solution to the first mixed problem for the telegraph equation with a nonlinear potential [12]; 3) the classical solution to the Cauchy–Darboux problem for a one-dimensional wave equation with power nonlinearity [6]; 4) the classical solution to the Goursat problem on a plane for a semilinear hyperbolic equation [13]. A rather interesting application of the method of successive approximations is shown in the work [14], where an economic problem is qualitatively solved. The problem under study, namely, the first mixed problem for one-dimensional mildly quasilinear wave equation in the first quadrant of the plane, was also studied by the method of successive approximations [15–18]. In this paper, we present a new approach to solving this problem based on the fixed-point principle. This fixed point principle, which is a generalization of Banach's theorem to the case of locally convex spaces, for the mixed problem in the first quadrant is applied for the first time in the present paper in the corresponding locally convex spaces.

2. Auxiliary material

In this section we present a generalization of Banach's theorem to the case of locally convex spaces, as well as all the necessary auxiliary definitions for this.

Definition 2.1. Let X be a locally convex space whose topology is determined by a system of seminorms $\{p_i\}_{i\in\mathfrak{I}}$. A mapping $f\colon X \mapsto X$ is called $\mathfrak{L}$-Lipschitz if $p_i(f(x_1), f(x_2)) \leqslant \mathfrak{L} p_i(x_1, x_2)$ for any $i \in \mathfrak{I}$, $x_1 \in X$, $x_2 \in X$.

Definition 2.2. Let X be a locally convex space whose topology is given by a system of seminorms $\{p_i\}_{i\in\mathfrak{I}}$. A mapping $f\colon X \mapsto X$ is called $\mathfrak{p}$-contracting if it is $\mathfrak{p}$-Lipschitz and $0 \leqslant \mathfrak{p} < 1$.

Remark 2.3. *In the literature, in a locally convex space X with a topology given by a system of seminorms $\{p_i\}_{i\in\mathfrak{I}}$, a mapping $f\colon X \mapsto X$ is called p-contracting for some $p \in \{p_i\}_{i\in\mathfrak{I}}$ if there exists a constant $k_p \in [0,1)$ such that $p(f(x_1), f(x_2)) \leqslant k_p p(x_1, x_2)$. However, our definition differs in that it establishes an explicit correspondence between $\mathfrak{L}$-Lipschitz and p-contracting maps, similar to the case of standard definitions for metric spaces.*

Theorem 2.4. *Let X be a sequentially complete Hausdorff locally convex space whose topology is determined by the system of seminorms $\{p_i\}_{i\in\mathfrak{I}}$ and let the mapping $f\colon X \mapsto X$ be $\mathfrak{p}$-contracting. Then the mapping f has a unique fixed point $x^* \in X$, and it is the limit of any sequence $x_{n+1} = f(x_n)$, where x_0 is any element of X.*

The **proof** is given in paper [19].

3. Statement of the problem

In the domain $Q = (0,\infty) \times (0,\infty)$ of two independent variables $(t,x) \in \overline{Q}$, we consider a one-dimensional nonlinear wave equation

$$\Box u(t,x) + f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x)) = F(t,x),\ (t,x) \in Q, \tag{1}$$

where $\Box = \partial_t^2 - a^2\partial_x^2$ is the d'Alembert operator ($a > 0$ for definiteness), F is a function given on the set $\overline{Q}$, and f is a function given on the set $\overline{Q} \times \mathbb{R}^3$. Equation (1) is equipped with the initial conditions

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \quad x \in [0,\infty), \tag{2}$$

and the boundary condition

$$u(t,0) = \mu(t), \quad t \in [0,\infty), \tag{3}$$

where φ, ψ, and μ are functions given on the half-line $[0,\infty)$.

Example 3.1. Setting $F \equiv 0$ and $f(t,x,u,u_t,u_x) = b\sin(\lambda u)$ in Eq. (1) yields the sine-Gordon equation, which has a number of applications in physics [20].

Example 3.2. Setting $F \equiv 0$ and $f(t,x,u,u_t,u_x) = \widetilde{f}(u)$ in Eq. (1) yields the nonlinear Klein–Gordon–Fock equation, which arises in differential geometry and various areas of physics (superconductivity, slip in crystals, waves in ferromagnets, laser pulses in two-phase media, etc.) [21].

Example 3.3. Setting $F \equiv 0$ and $f(t,x,u,u_t,u_x) = \varepsilon^{-2}\big((1+\widetilde{g}(u))u_t - \widetilde{f}(u)\big)$, $\varepsilon > 0$ in Eq. (1) yields a nonlinear equation describing the voltage on electrical communication lines with nonlinear shunt conductivity and a series connection of an inductive load, the movement and reproduction of tissue cells and unicellular organisms, and a branching random walk [22].

Example 3.4. Setting $F \equiv 0$ and $f(t,x,u,u_t,u_x) = \sinh(u)$ in Eq. (1) yields the sinh-Gordon equation. This equation is a special case of the Toda chain and can model the interaction between neighboring particles of the same mass in crystal lattices [23].

4. Integro-differential equation

We divide the domain Q by the characteristic $x - at = 0$ into two subdomains $Q^{(j)} = \{(t,x) : (-1)^j(at-x) > 0\}$, $j = 1,2$.

Let us consider the following coupled equations

$$u^{(1)}(t,x) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at}\psi(\xi)d\xi + \frac{1}{4a^2}\int\limits_{x-at}^{x+at}dz\int\limits_{x-at}^{z}\Bigg[F\left(\frac{z-y}{2a},\frac{z+y}{2}\right) - f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\partial_t u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\partial_x u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\Bigg]dy, \quad (t,x) \in \overline{Q^{(1)}}, \tag{4}$$

$$u^{(2)}(t,x) = \mu\left(t-\frac{x}{a}\right) - u^{(1)}\left(\frac{at-x}{2a},\frac{at-x}{2}\right) + u^{(1)}\left(\frac{x+at}{2a},\frac{x+at}{2}\right) - \frac{1}{4a^2}\times \int\limits_{at-x}^{x+at}dz\int\limits_{0}^{x-at}\Bigg[F\left(\frac{z-y}{2a},\frac{y+z}{2}\right) - f\left(\frac{z-y}{2a},\frac{y+z}{2},u^{(2)}\left(\frac{z-y}{2a},\frac{y+z}{2}\right),\partial_t u^{(2)}\left(\frac{z-y}{2a},\frac{y+z}{2}\right),\partial_x u^{(2)}\left(\frac{z-y}{2a},\frac{y+z}{2}\right)\right)\Bigg]dy, \quad (t,x) \in \overline{Q^{(2)}}. \tag{5}$$

On the closure $\overline{Q}$ of the domain Q, we define a function u as the one coinciding with the solution $u^{(j)}$ of Eqs. (4) and (5)

$$u(t,x) = u^{(j)}(t,x), \quad (t,x) \in \overline{Q^{(j)}}, \quad j = 1,2, \tag{6}$$

on the closure $\overline{Q^{(j)}}$ of the domain $Q^{(j)}$. The following theorem holds.

Theorem 4.1. *Let the conditions $f \in C^1(\overline{Q}\times\mathbb{R}^3)$, $F \in C^1(\overline{Q})$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, and $\mu \in C^2([0,\infty))$ be satisfied. A function u from the class $C^2(\overline{Q})$ is a solution of the mixed problem* (1)–(3) *if it can be represented in the form* (4)–(6) *and the matching conditions*

$$\mu(0) = \varphi(0), \tag{7}$$

$$\mu'(0) = \psi(0), \tag{8}$$

$$\mu''(0) = F(0,0) - f(0,0,\varphi(0),\psi(0),\varphi'(0)) + a^2\varphi''(0) \tag{9}$$

are satisfied.

The **proof** is presented in works [15; 18].

For brevity of further reasoning, we will rewrite Eqs. (4) and (5) in the form

$$u^{(1)}(t,x) = K_1[u^{(1)}](t,x) = G_1(t,x) - \frac{1}{4a^2}\int\limits_{x-at}^{x+at}dz\int\limits_{x-at}^{z} f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\partial_t u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\partial_x u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)dy, \quad (t,x) \in \overline{Q^{(1)}}, \tag{10}$$

$$u^{(2)}(t,x)=K_2[u^{(2)}](t,x)=G_2(t,x)+\frac{1}{4a^2}\int\limits_{at-x}^{x+at}dz\int\limits_{0}^{x-at}f\left(\frac{z-y}{2a},\frac{y+z}{2},u^{(2)}\left(\frac{z-y}{2a},\frac{y+z}{2}\right),\right.$$
$$\left.\partial_t u^{(2)}\left(\frac{z-y}{2a},\frac{y+z}{2}\right),\partial_x u^{(2)}\left(\frac{z-y}{2a},\frac{y+z}{2}\right)\right)dy,\quad (t,x)\in\overline{Q^{(2)}}, \tag{11}$$

and introduce the sets $Q_T=Q\cap\{(t,x)\mid t\leqslant T\}$, $Q_T^{(j)}=Q^{(j)}\cap\{(t,x)\mid t\leqslant T\}$, $j=1,2$, $Q_n^*=$ $=[nT,(n+1)T]\times[0,\infty)$, $n\in\mathbb{N}\cup\{0\}$.

Assume that the topology of the Fréchet space $C^1\left(\overline{Q_T^{(1)}}\right)$ is defined by a countable system of seminorms $\{\|\bullet\|_{C^1(\Omega_m)}\}_{m=\lceil 2aT+1\rceil}^{\infty}$, where

$$\Omega_m=\mathrm{Conv}\{(0,0),(0,m),(T,aT),(T,m-aT)\}.$$

This topology is well defined because

$$\bigcup_{m=\lceil 2aT+1\rceil}^{\infty}\mathrm{Conv}\{(0,0),(0,m),(T,aT),(T,m-aT)\}=\overline{Q_T^{(1)}}.$$

The following lemma holds.

Lemma 4.2. *Let the conditions $G_1\in C^1(\overline{Q_T^{(1)}})$ and $f\in C(\overline{Q_T^{(1)}}\times\mathbb{R}^3)$ be satisfied and let the function f satisfy the Lipschitz condition*

$$|f(t,x,z_1,z_2,z_3)-f(t,x,w_1,w_2,w_3)|\leqslant L(|z_1-w_1|+|z_2-w_2|+|z_3-w_3|) \tag{12}$$

in the last three variables with constant L. Then the operator $K_1\colon C^1(\overline{Q_T^{(1)}})\mapsto C^1(\overline{Q_T^{(1)}})$, acting by formula (10)*, is $\mathfrak{L}$-Lipschitz, where $\mathfrak{L}=3L\max\{T,T^2\}\max\{1,a^{-1}\}$.*

Proof. Let us find the norm estimates

$$\|K_1[u]-K_1[\widetilde{u}]\|_{C(\Omega_m)}=$$
$$=\max_{(t,x)\in\Omega_m}\left|\frac{1}{4a^2}\int\limits_{x-at}^{x+at}dz\int\limits_{x-at}^{z}\left(f\left(\frac{z-y}{2a},\frac{z+y}{2},u\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\partial_t u\left(\frac{x-y}{2a},\frac{z+y}{2}\right),\right.\right.\right.$$
$$\left.\partial_x u\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)-f\left(\frac{z-y}{2a},\frac{z+y}{2},\widetilde{u}\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\partial_t\widetilde{u}\left(\frac{z-y}{2a},\frac{z+y}{2}\right),\right.$$
$$\left.\left.\left.\partial_x u\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\right)dy\right|\leqslant$$
$$\leqslant\frac{1}{4a^2}\int\limits_{x-at}^{x+at}dz\int\limits_{x-at}^{z}L(|u-\widetilde{u}|+|\partial_t u-\partial_t\widetilde{u}|+|\partial_x u-\partial_x\widetilde{u}|)\left(\frac{z-y}{2a},\frac{z+y}{2}\right)dy\leqslant$$
$$\leqslant\frac{1}{2}LT^2\left(\|u-\widetilde{u}\|_{C(\Omega_m)}+\|\partial_t u-\partial_t\widetilde{u}\|_{C(\Omega_m)}+\|\partial_x u-\partial_x\widetilde{u}\|_{C(\Omega_m)}\right).$$

Similarly, we obtain

$$\|\partial_t K_1[u]-\partial_t K_1[\widetilde{u}]\|_{C(\Omega_m)}\leqslant$$
$$\leqslant LT\left(\|u-\widetilde{u}\|_{C(\Omega_m)}+\|\partial_t u-\partial_t\widetilde{u}\|_{C(\Omega_m)}+\|\partial_x u-\partial_x\widetilde{u}\|_{C(\Omega_m)}\right),$$
$$\|\partial_x K_1[u]-\partial_x K_1[\widetilde{u}]\|_{C(\Omega_m)}\leqslant$$
$$\leqslant\frac{LT}{a}\left(\|u-\widetilde{u}\|_{C(\Omega_m)}+\|\partial_t u-\partial_t\widetilde{u}\|_{C(\Omega_m)}+\|\partial_x u-\partial_x\widetilde{u}\|_{C(\Omega_m)}\right).$$

Since $\|\bullet\|_{C^1(\Omega_m)}=\|\bullet\|_{C(\Omega_m)}+\|\partial_t\bullet\|_{C(\Omega_m)}+\|\partial_x\bullet\|_{C(\Omega_m)}$, introducing the notation $\widetilde{T}=\max\{T,T^2\}$, $A=$ $=\max\{1,a^{-1}\}$, we obtain

$$\|K_1[u]-K_1[\widetilde{u}]\|_{C(\Omega_m)}\leqslant L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\Omega_m)},$$
$$\|\partial_t K_1[u]-\partial_t K_1[\widetilde{u}]\|_{C(\Omega_m)}\leqslant L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\Omega_m)},$$

$$\|\partial_x K_1[u] - \partial_x K_1[\widetilde{u}]\|_{C(\Omega_m)} \leqslant L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\Omega_m)},$$

where u и $\widetilde{u}$ are arbitrary functions from the space $C^1(\overline{Q_T^{(1)}})$. So,

$$\|K_1[u] - K_1[\widetilde{u}]\|_{C^1(\Omega_m)} \leqslant 3L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\Omega_m)}.$$

With respect to any of the seminorms, that defines the topology of the Fréchet space $C^1\left(\overline{Q_T^{(1)}}\right)$, the operator K satisfies the Lipschitz condition with constant $3L\max\{T,T^2\}\max\{1,a^{-1}\}$. □

Corollary 4.3. *Let the conditions $G_1 \in C^1(\overline{Q_T^{(1)}})$ and $f \in C(\overline{Q_T^{(1)}} \times \mathbb{R}^3)$ be satisfied, let the function f satisfy the Lipschitz condition* (12)*, and let $T < \min\left\{1, \dfrac{1}{3L\max\{1,a^{-1}\}}\right\}$. Then the operator $K_1 \colon C^1(\overline{Q_T^{(1)}}) \mapsto C^1(\overline{Q_T^{(1)}})$, acting by formula* (10)*, is $\mathfrak{p}$-contracting, where $\mathfrak{p} = \dfrac{1}{3L\max\{1,a^{-1}\}}$.*

By combining Corollary 4.3 with Theorem 2.4, we obtain the following statement.

Corollary 4.4. *Let the conditions $G_1 \in C^1(\overline{Q_T^{(1)}})$ and $f \in C(\overline{Q_T^{(1)}} \times \mathbb{R}^3)$, let the function f satisfy the Lipschitz condition* (12)*, and let $T < \min\left\{1, \dfrac{1}{3L\max\{1,a^{-1}\}}\right\}$. Then there exists a unique solution $u^{(1)}$ of Eq.* (10) *in the class $C^1(\overline{Q_T^{(1)}})$.*

Lemma 4.5. *Let the conditions $G_2 \in C^1(\overline{Q_T^{(2)}})$ and $f \in C(\overline{Q_T^{(2)}} \times \mathbb{R}^3)$ be satisfied and let the function f satisfy the Lipschitz condition* (12)*. Then the operator $K_2 \colon C^1(\overline{Q_T^{(2)}}) \mapsto C^1(\overline{Q_T^{(2)}})$, acting by formula* (11)*, is $\mathfrak{L}$-Lipschitz, where $\mathfrak{L} = 3L\max\{T,T^2\}\max\{1,a^{-1}\}$.*

Proof. Similar to Lemma 4.2, we introduce the notation $\widetilde{T} = \max\{T,T^2\}$, $A = \max\{1,a^{-1}\}$ and calculate

$$\|K_2[u] - K_2[\widetilde{u}]\|_{C(\overline{Q_T^{(2)}})} \leqslant \frac{LT^2\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})}}{2} \leqslant L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})},$$
$$\|\partial_t K_2[u] - \partial_t K_2[\widetilde{u}]\|_{C(\overline{Q_T^{(2)}})} \leqslant LT\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})} \leqslant L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})},$$
$$\|\partial_x K_2[u] - \partial_x K_2[\widetilde{u}]\|_{C(\overline{Q_T^{(2)}})} \leqslant \frac{LT\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})}}{a} \leqslant L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})}.$$

where u and $\widetilde{u}$ are arbitrary functions from the space $C^1(\overline{Q_T^{(2)}})$. Therefore,

$$\|K[u] - K[\widetilde{u}]\|_{C^1(\overline{Q_T^{(2)}})} \leqslant 3L\widetilde{T}A\|u-\widetilde{u}\|_{C^1(\overline{Q_T^{(2)}})}.$$

□

Corollary 4.6. *Let the conditions $G_2 \in C^1(\overline{Q_T^{(2)}})$, $f \in C(\overline{Q_T^{(2)}} \times \mathbb{R}^3)$ be satisfied, let the function f satisfy the Lipschitz condition* (12)*, and let $T < \min\left\{1, \dfrac{1}{3L\max\{1,a^{-1}\}}\right\}$. Then the operator $K_2 \colon C^1(\overline{Q_T^{(2)}}) \mapsto C^1(\overline{Q_T^{(2)}})$, acting by formula* (11)*, is contractive.*

Using the Banach fixed-point theorem, we obtain the following corollary.

Corollary 4.7. *Let the conditions $G_2 \in C^1(\overline{Q_T^{(2)}})$ and $f \in C(\overline{Q_T^{(2)}} \times \mathbb{R}^3)$ be satisfied, let the function f satisfy the Lipschitz condition* (12)*, and let $T < \min\left\{1, \dfrac{1}{3L\max\{1,a^{-1}\}}\right\}$. Then there exists a unique solution $u^{(2)}$ of Eq.* (11) *in the class $C^1(\overline{Q_T^{(2)}})$.*

The following theorem on the smoothness of solutions Eqs. (10) and (11) hold.

Theorem 4.8. *Let the conditions* $F \in C^1(\overline{Q_T})$, $f \in C^1(\overline{Q_T} \times \mathbb{R}^3)$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, *and* $\mu \in C^2([0,T])$ *be satisfied, and let the functions* $u^{(1)} \in C^1(\overline{Q_T^{(1)}})$ *and* $u^{(2)} \in C^1(\overline{Q_T^{(2)}})$ *solve Eqs.* (10) *and* (11)*, respectively. Then the function*

$$u_T(t,x) = \begin{cases} u^{(1)}(t,x) & (t,x) \in \overline{Q_T^{(1)}}, \\ u^{(2)}(t,x) & (t,x) \in \overline{Q_T^{(2)}}, \end{cases}$$

belongs to the class $C^2(\overline{Q_T})$ *if and only if the matching conditions* (7)–(9) *are fulfilled.*

The **proof** is presented in works [15; 18].

5. Local classical solution

According to Theorem 4.1 and Corollaries 4.4 and 4.7, under the smoothness conditions $F \in C^1(\overline{Q_T})$, $f \in C^1(\overline{Q_T} \times \mathbb{R}^3)$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, and $\mu \in C^2([0,T])$, the Lipschitz condition (12), and the matching conditions (7)–(9), we constructed the classical solution u of the problem (1)–(3) on the set $\overline{Q_T}$, where T can be taken as $T = \frac{1}{2}\min\left\{1, \frac{1}{3L\max\{1,a^{-1}\}}\right\}$. This solution is determined by formulas (4)–(6). Furthermore, conditions (7)–(9) are sufficient and necessary.

6. Global classical solution

It turns out that using the conjugation conditions

$$u_1(T,x) = u_0(T,x), \quad \partial_t u_1(T,x) = \partial_t u_0(T,x), \quad x \in [0,\infty), \tag{13}$$

we can continue the solution u_0 of the first mixed problem (1)–(3) to the set $\overline{Q_1^*}$. Since the functions $x \mapsto u_0(T,x)$ and $x \mapsto \partial_t u_1(T,x)$ belong to the classes $C^2([0,\infty))$ and $C^1([0,\infty))$, respectively, the matching conditions are satisfied

$$\begin{aligned} \mu(T) &= u_0(T,0), \\ \mu'(T) &= \partial_t u_0(T,0), \\ \mu''(T) &= F(0,0) - f(0,0,u_0(T,0),\partial_t u_0(T,0),\partial_x u_0(T,0)) + a^2\partial_x^2 u_0(T,0). \end{aligned}$$

The function $u_1 \colon \overline{Q_1^*} \ni (t,x) \mapsto u_1(t,x) \in \mathbb{R}$ will belong to the class $C^2(\overline{Q_1^*})$ if, for example, $F \in C^1(\overline{Q})$, $f \in C^1(\overline{Q} \times \mathbb{R}^3)$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, and $\mu \in C^2([0,\infty))$.

Differentiating equalities (13) with respect to x yields the following:

$$\begin{aligned} &\partial_x u_1(T,x) = \partial_x u_0(T,x), \quad \partial_x^2 u_1(T,x) = \partial_x^2 u_0(T,x), \\ &\partial_x\partial_t u_1(T,x) = \partial_x\partial_t u_0(T,x), \quad x \in [0,\infty). \end{aligned} \tag{14}$$

From Eq. (1) we express the quantities $\partial_t^2 u_j(T,x)$, $j = 0,1$,

$$\partial_t^2 u_j(T,x) = a^2\partial_x^2 u_j(T,x) + F(T,x) - f(T,x,u_j(T,x),\partial_t u_j(T,x),\partial_x u_j(T,x)), \quad x \in [0,\infty). \tag{15}$$

Due to expressions (13) and (14) and the continuity of the functions f and F in expression (15), the right-hand sides are equal for $j = 0$ and $j = 1$. Thus, the left-hand sides are also equal, i. e.,

$$\partial_t^2 u_1(T,x) = \partial_t^2 u_0(T,x), \qquad x \in [0,\infty). \tag{16}$$

Conditions (13), (14), and (16) mean that the function

$$u_{0,1}(t,x) = \begin{cases} u_0(t,x), & (t,x) \in [0,T] \times [0,\infty), \\ u_1(t,x), & (t,x) \in [T,2T] \times [0,\infty), \end{cases}$$

belongs to the class $C^2([0,2T] \times [0,\infty))$ and satisfies equation (1) on the set $(0,2T) \times [0,\infty)$. Note that a different choice of the matching conditions (13) will result in at least one of the functions $u_{0,1}$ or $\partial_t u_{0,1}$ being discontinuous, which will entail $u_{0,1} \notin C^2([0,2T] \times [0,\infty))$.

Again, for the function u_1 the following conditions hold:

$$\mu(2T) = u_1(2T,0),$$
$$\mu'(2T) = \partial_t u_1(2T,0),$$
$$\mu''(2T) = F(0,0) - f(0,0,u_1(2T,0),\partial_t u_1(2T,0),\partial_x u_1(2T,0)) + a^2\partial_x^2 u_1(2T,0),$$

which, together with the smoothness conditions of $F \in C^1(\overline{Q_T})$, $f \in C^1(\overline{Q_T} \times \mathbb{R}^3)$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$ and $\mu \in C^2[0,\infty))$, the matching conditions (7)–(9) and the Lipschitz condition of f with respect to the last three variables, makes it possible to extend the solution to the set $\overline{Q_2^*}$ using a similar scheme. It proves the base of induction.

Suppose that after the n-th step, we have a function

$$u_{0,n}(t,x) = u_i(t,x), \quad (t,x) \in \overline{Q_i^*},$$

which is defined on the set $[0,(n+1)T] \times [0,\infty)$, belongs to the class $C^2([0,(n+1)T] \times [0,\infty))$, is a solution to problem (1)–(3) on this set, and satisfies the conditions

$$\mu((n+1)T) = u_{0,n}((n+1)T,0),$$
$$\mu'((n+1)T) = \partial_t u_{0,n}((n+1)T,0),$$
$$\mu''((n+1)T) = F(0,0) - f(0,0,u_{0,n}((n+1)T,0),\partial_t u_{0,n}((n+1)T,0),$$
$$\partial_x u_{0,n}((n+1)T,0)) + a^2\partial_x^2 u_{0,n}((n+1)T,0).$$

We extend this function to the set $\overline{Q_{n+1}^*}$ using the conditions

$$u_n((n+1)T,x) = u_{0,n}((n+1)T,x), \quad \partial_t u_n((n+1)T,x) = \partial_t u_{0,n}((n+1)T,x), \quad x \in [0,\infty).$$

Based on formulas (4) and (5), we obtain

$$\mu((n+2)T) = u_n((n+2)T,0),$$
$$\mu'((n+2)T) = \partial_t u_n((n+2)T,0),$$
$$\mu''((n+2)T) = F(0,0) - f(0,0,u_n((n+2)T,0),\partial_t u_n((n+2)T,0),$$
$$\partial_x u_n((n+2)T,0)) + a^2\partial_x^2 u_n((n+2)T,0),$$

Using the above-proposed scheme, we prove that the function $u_{0,n+1}$ belongs to the class $C^2([0,(n+2)T] \times [0,\infty))$. Thus, the induction step is proved.

Thus, a global classical solution of the problem was constructed under certain smoothness conditions, matching conditions, and Lipschitz conditions. However, condition (12) can be slightly weakened to:

$$|f(t,x,z_1,z_2,z_3) - f(t,x,w_1,w_2,w_3)| \leqslant C_{\text{lip}}(t,x)(|z_1-w_1| + |z_2-w_2| + |z_3-w_3|), \tag{17}$$

where C_{lip} is some continuous function. We show that in this case one can also construct a unique global classical solution if we impose the following smoothness conditions $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, $\mu \in C^2([0,\infty))$, $f \in C^1(\overline{Q} \times \mathbb{R}^3)$, $F \in C^1(\overline{Q})$ and the matching conditions (7)–(9). Note that the fulfillment of condition (17) for all $(t,x) \in \Theta_m = \text{Conv}\{(0,0),(0,m),(m/(2a),m/2),(m/(2a),0)\}$, $m \in \mathbb{N}$, implies inequality (12) with constant $L = \|C_{\text{lip}}\|_{C(\Theta_m)}$ on the same set $(t,x) \in \Theta_m$. We define the function $v^{(m)} \colon \Theta_m \ni (t,x) \mapsto v^{(m)}(t,x) \in \mathbb{R}$ as the classical solution of the first mixed problem (1)–(3) on the set Θ_m. According to the approach described earlier, such a solution exists and is unique.

We extend $v^{(m)}$ to the entire set $\overline{Q}$ so that $v^{(m)} \in C^2(\overline{Q})$. We claim that the function $u^{(\infty)} = \lim\limits_{m\to\infty} v^{(m)}$ is a classical solution of the problem (1)–(3) on the set $\overline{Q}$. Note that the topology of the Fréchet space $C^2(\overline{Q})$ can be defined by a countable system of seminorms $\mathcal{P} = (\mathfrak{p}_i)_{i=1}^{\infty}$, where $\mathfrak{p}_i(u) = \|u\|_{C^2(\Theta_i)}$, since $\bigcup\limits_{i=1}^{\infty} \Theta_i = \overline{Q}$. Since for any natural numbers m and j the equality $\left[v^{(m+j)} - v^{(m)}\right]\big|_{\Theta_m} \equiv 0$ holds, the sequence $v^{(1)}, v^{(2)}, \ldots$ converges with respect to any seminorm $\mathfrak{p} \in \mathcal{P}$. This implies the existence and uniqueness of the limit $\lim\limits_{m\to\infty} v^{(m)}$ in the space $C^2(\overline{Q})$. Next, we must verify that the function $u^{(\infty)}$ satisfies Eq. (1), the initial (2) and the boundary conditions (3). Indeed, since for any fixed point $(t,x) \in \overline{Q}$ there exists a natural number m such that $u^{(\infty)}(t,x) = v^{(m)}(t,x)$, the function $u^{(\infty)}$ satisfies Eq. (1) at the point (t,x), and since the point $(t,x) \in \overline{Q}$ is arbitrary, it satisfies the equation in the entire set $\overline{Q}$. The satisfaction of conditions (2) and (3) is similarly proved. The uniqueness of the global solution is proved by contradiction. Assume that there exist two global solutions $u^{(\infty,1)}$ and $u^{(\infty,2)}$. Then, for a fixed $m \in \mathbb{N}$, consider their restrictions $u^{(\infty,1)}|_{\Theta_m}$ and $u^{(\infty,2)}|_{\Theta_m}$, which are solutions of the first mixed problem (1)–(3) on the set Θ_m. Since the solution to this problem is unique, $u^{(\infty,1)}|_{\Theta_m} \equiv u^{(\infty,2)}|_{\Theta_m}$. From this, we obtain $\left[u^{(\infty,1)} - u^{(\infty,2)}\right]\big|_{\Theta_m} \equiv 0$. Since m is arbitrary and $\bigcup\limits_{i=1}^{\infty} \Theta_i = \overline{Q}$, $u^{(\infty,1)} - u^{(\infty,2)} \equiv 0$ on the set $\overline{Q}$, which proves uniqueness. We state the result as the following theorem.

Theorem 6.1. *Let the conditions* $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, $\mu \in C^2([0,\infty))$, $f \in C^1(\overline{Q} \times \mathbb{R}^3)$, *and* $F \in C^1(\overline{Q})$ *be satisfied and let the function* f *satisfy the Lipschitz condition* (17) *with the function* $C_{\text{lip}} \in C(\overline{Q})$. *The first mixed problem* (1)–(3) *has a unique solution* u *from the class* $C^2(\overline{Q})$ *if and only if the conditions* (7)–(9) *are satisfied. This solution has the representation* (4)–(6).

The **proof** follows from the above reasoning.

Remark 6.2. *Formulas* (4)–(6), *which define the equations for the classical solution* u *of the problem* (1)–(3), *can be rewritten in a simpler form, similar to that given in* [24, p. 76–77], *namely:*

$$u(t,x) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(\xi)\,d\xi + \\ + \frac{1}{2a}\int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \left(F(\tau,\xi) - f(\tau,\xi,u(\tau,\xi),\partial_t u(\tau,\xi),\partial_x u(\tau,\xi))\right)d\xi, \quad (t,x) \in \overline{Q^{(1)}}, \tag{18}$$

$$u(t,x) = \mu\left(t - \frac{x}{a}\right) + \frac{\varphi(x+at)-\varphi(at-x)}{2} + \frac{1}{2a}\int\limits_{at-x}^{x+at} \psi(\xi)\,d\xi + \\ + \frac{1}{2a}\int\limits_0^t d\tau \int\limits_{|x-a(t-\tau)|}^{x+a(t-\tau)} \left(F(\tau,\xi) - f(\tau,\xi,u(\tau,\xi),\partial_t u(\tau,\xi),\partial_x u(\tau,\xi))\right)d\xi, \quad (t,x) \in \overline{Q^{(2)}}. \tag{19}$$

Proof. Indeed, by changing the variables

$$\tau = \frac{z-y}{2a}, \quad \xi = \frac{z+y}{2}$$

in the integrals in formulas (4) and (5), we arrive at expressions (18) and (19). □

Remark 6.3. *The condition* $F \in C^1(\overline{Q})$ *of Theorems* 4.1 *and* 6.1 *can be replaced with*

$$F \in C(\overline{Q}), \quad \left(\overline{Q} \ni (t,x) \mapsto \int\limits_0^t F(\tau,|x \pm a(t-\tau)|)\,d\tau \in \mathbb{R}\right) \in C^1(\overline{Q}). \tag{20}$$

Proof. If condition (20) holds, then the right-hand sides of expressions (18) and (19) represent a function u from the space $C^2(\overline{Q})$ under conditions $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, $\mu \in C^2([0,\infty))$, $f \in C^1(\overline{Q} \times \mathbb{R}^3)$, and (7)–(9) [25; 26]. □

Remark 6.4. *If the function* F *has the form* $F(t,x) = F(t)$ *or* $F(t,x) = F(x)$ *then the condition* $F \in C^1(\overline{Q})$ *of Theorems* 4.1 *and* 6.1 *can be replaced with* $F \in C(\overline{Q})$.

Proof. Let us first consider the case when $F(t,x) = F(t)$. We have

$$\left(\overline{Q} \ni (t,x) \mapsto \int\limits_0^t F(\tau,|x \pm a(t-\tau)|)\,d\tau = \int\limits_0^t F(\tau)\,d\tau \in \mathbb{R}\right) \in C^1(\overline{Q}).$$

The case $F(t,x) = F(t)$ is proved. Now consider the case $F(t,x) = F(x)$. We have

$$\int\limits_0^t F(\tau,|x \pm a(t-\tau)|)\,d\tau = \int\limits_0^t F(|x \pm a(t-\tau)|)\,d\tau.$$

By virtue of the formulas

$$\frac{\partial}{\partial t}\left(\int_0^t F(x+a(t-\tau))\,d\tau\right) = F(x) + \int_0^t aF'(x+a(t-\tau))\,d\tau = F(x+at),$$

$$\frac{\partial}{\partial x}\left(\int_0^t F(x+a(t-\tau))\,d\tau\right) = \int_0^t F'(x+a(t-\tau))\,d\tau = \frac{F(x+at)-F(x)}{a},$$

we conclude that

$$\left(\overline{Q} \ni (t,x) \mapsto \int_0^t F(\tau, x+a(t-\tau))\,d\tau \in \mathbb{R}\right) \in C^1(\overline{Q}).$$

Similary, we derive that

$$\left(\overline{Q} \ni (t,x) \mapsto \int_0^t F(\tau, |x-a(t-\tau)|)\,d\tau \in \mathbb{R}\right) \in C^1(\overline{Q}).$$

Thus, the case $F(t,x) = F(x)$ is also proved. □

Conclusions

In this paper, the first mixed problem for a mildly quasilinear one-dimensional wave equation in the first quadrant has been studied in the classical formulation. The problem contains Cauchy conditions on the spatial semi-axis and a Dirichlet condition on the temporal semi-axis, with nonlinearity depending on the unknown function and its first-order derivatives.

The main contributions of the work are as follows:

1. Integral representation of the solution. The solution is constructed implicitly through a system of coupled integro-differential equations, derived by partitioning the domain along the characteristic line $x - at = 0$. This representation generalizes the classical d'Alembert formula to the nonlinear case and naturally incorporates the initial and boundary data.

2. Fixed-point approach in locally convex spaces. The solvability of the integro-differential system is established using a generalization of the Banach fixed-point theorem to sequentially complete Hausdorff locally convex spaces. This allows the treatment of the nonlinear terms under a Lipschitz condition, ensuring the existence and uniqueness of local classical solutions.

3. Local and global classical solvability. Under smoothness assumptions on the data and nonlinearity, and subject to matching conditions at the origin, the existence of a unique classical solution is proved locally in time. Using a step-by-step extension method based on conjugation conditions, the local solution is extended globally to the entire first quadrant, preserving C^2-regularity.

4. Weakened Lipschitz and regularity conditions. Results extended to a spatially and time dependent Lipschitz condition, broadening the class of admissible nonlinearities. Additionally, regularity requirements on the nonhomogeneous term F are relaxed, allowing for certain integrability conditions rather than pointwise smoothness.

5. Physical and mathematical relevance. The framework encompasses several important equations of mathematical physics, including the sine-Gordon, double sine-Gordon, and nonlinear Klein–Gordon–Fock equations, demonstrating the applicability of the results to problems in wave propagation, differential geometry, and nonlinear dynamics.

In summary, the paper provides a rigorous and constructive treatment of the first mixed problem for a mildly quasilinear wave equation, combining classical PDE techniques with functional-analytic methods. The results extend prior works on semilinear and quasilinear wave equations and offer a versatile approach for studying nonlinear initial-boundary value problems in unbounded domains.

Funding

The work was supported by state program for scientific research “Interdisciplinary and Synergetic Research” (“Convergence-2030”), subprogram “Modern Mathematical Methods and Their Applications”, task 1.05 “Classical Solutions, Development of New Methods for Studying Problems of the Theory of Partial Differential Equations”, R&D 1.05.1 “Classical Methods for Solving and Proving the Hadamard Well-Posedness of Problems for Partial Differential Equations”.

Declaration of AI

DeepSeek was used to improve the quality of the text while working on the article. The main prompts were: correct grammar, proofread the text, write conclusions.

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